

LINEAR SELECTIONS OF SUPERLINEAR SET-VALUED MAPS WITH SOME APPLICATIONS TO ANALYSIS

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ABSTRACT. A. Ya. Zaslavskii's results [48] on the existence of a linear (affine) selection for a linear (affine) or superlinear (convex) map $\Phi : K \rightarrow 2^Y$ defined on a convex cone (convex set) K having the interpolation property are extended. We prove that they hold true under more general conditions on the values of the mapping and study some other properties of the selections. This leads to a characterization of Choquet simplexes in terms of the existence of continuous affine selections for arbitrary continuous convex maps. A few applications to analysis are given, including a construction that leads to the existence of a (not necessarily bounded) solution for the corona problem in polydisk \mathbb{D}^n with radial boundary values that are bounded almost everywhere on \mathbb{T}^n .

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0. INTRODUCTION

We begin by a cursory introduction to the subject of this paper before going into formal details. Suppose that $T : K \rightarrow \mathcal{V} \subset 2^Y$ is a superadditive set-valued map, i. e. that $T(x) + T(y) \subset T(x + y)$ for all x and y (such maps are also known as convex processes; see, e. g., [31]). When does T have a linear selection S , meaning that $S(x) \in T(x)$ for all x , and what is the set of all such selections? There is a closely related question about affine selections of convex maps, i. e. maps satisfying $(1 - \theta)T(x) + \theta T(y) \subset T((1 - \theta)x + \theta y)$ for all $0 < \theta < 1$ and x, y . This question appears to be more intuitive than its counterpart about superlinear maps (see Figure 1 below). Superlinear set-valued maps appear naturally in certain problems of economics among other things (see, e. g., [32], [4]).

The aim of this paper is to give these questions proper treatment and show how it leads to some useful tools that can be applied to certain problems in analysis. Our starting point is a remarkable work [48] by A. Ya. Zaslavskii in which he proved that a superlinear map T has a linear selection if the following conditions are sufficient: T is positive homogeneous, K is a cone with the so-called interpolation property (also known as the Riesz decomposition property) and \mathcal{V} is a collection of convex sets that has certain compactness properties. Similar results were obtained independently in [40] and [41] without positive homogeneity of T but under the assumption that K is a cone with a cone-basis, which is in general considerably weaker than the interpolation property. There are some results for sublinear and concave maps as well as superlinear maps; see, e. g., [30] and references contained therein. V. V. Gorokhovik obtained in [15] some interesting results establishing (under some restrictions) a natural relationship between an affine set-valued map and the set of its linear selections (see Figure 1).

All of the results related to linear selections of superlinear maps and convex selections of affine maps mentioned up to this point are obtained under some typical restrictions that allow fairly easy proofs, leaving the necessity of these restrictions largely unexplored. Although it appears to be difficult to characterize the exact necessary and sufficient conditions in the general case for results of this sort, in this paper we will study this question to an extent for various cases and obtain some generalizations. Perhaps the most significant improvement is that we

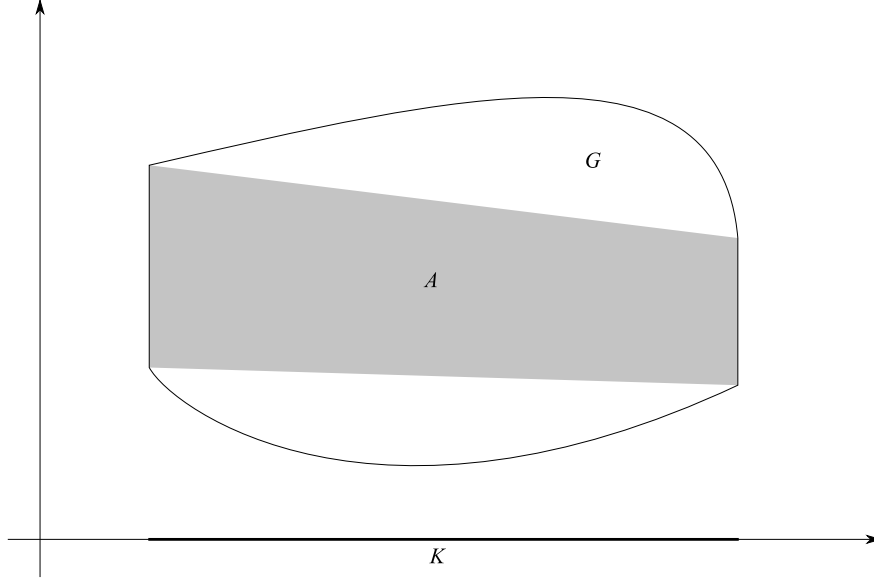


FIGURE 1. An example of the graph G of a convex map defined on a segment $K \subset \mathbb{R}$. Gray area A is the graph of the maximal affine submap of this map. The graph of any affine selection of G is contained within A and the collection of all such graphs covers the set A .

consider maps taking values in *sets of compact type* (i. e. in a collection of sets closed under intersection of centered families; see Section 4 for the definition) instead of the usual compact sets, which allows us to include maps taking values in convex closed sets in a Banach ideal lattice on \mathbb{Z} having the Fatou property that are closed in measure. Moderate refinement of the method of [48] leads to an exact condition on points x and y that guarantees the existence of a linear selection S satisfying $Sx = y$ for a given superlinear set-valued map T . In Section 7 we further characterize the sets of affine selections of a convex map under some restrictions and show, as a fairly easy consequence of the results presented in this paper, that given a compact convex set K in a locally convex linear topological space the following conditions are equivalent.

- (1) Every convex map on K taking values in bounded segments of the line \mathbb{R} has an affine selection.
- (2) K is a Choquet simplex.

The compact type condition cannot be dropped in general, as will be evident in Section 11. However, in the case of a cone having a cone-basis (see definition in Section 10), which in finite-dimensional case is any cone having a simplex as a base, it is not needed. Although the

assumption of the existence of a cone-basis is much stronger than the Riesz decomposition property, this result has an interesting by-product: in Section 11 we show that for no infinite-dimensional normed Banach ideal lattices of measurable functions the cone of nonnegative functions has a cone-basis.

The paper is organized as follows. In Section 1 we introduce basic definitions and discuss some simple properties. In Section 2 Zaslavskii's result concerning additive submaps of superadditive maps is presented. Section 3 contains some definitions and facts mostly of topological nature that will be used only sparingly, mostly for topological refinements of various results. In Section 4 main results about linear selections of superlinear and linear set-valued maps are presented. Proofs of the main theorem concerning linear selections of linear maps are given in Section 5. In Section 6 we establish the link between linear selections of superlinear maps and affine selections of convex maps, which allows us to transfer the results of Section 4 directly to the latter case. Section 7 is devoted to a characterization of Choquet simplexes in terms of the existence of certain linear selections. In Section 8 we provide a description of all continuous affine selections of an upper semicontinuous convex map defined on a metrizable Bauer simplex and acting into closed subsets of a Banach space and show that this method essentially characterizes only linear selections of superlinear maps acting on the set of Baire measures on a metrizable compact set. In Section 9 we show that the main results of this paper are applicable to maps taking values in convex sets that are closed in measure in a Banach ideal lattice on \mathbb{Z} satisfying the Fatou property. Section 10 contains simple versions of the main results for cones having a cone-basis. Although it builds on the terminology introduced up to that point, this section is essentially independent of the other sections. In Section 11 we give an example showing that in a core result about the existence of linear selections for linear set-valued maps the condition that the values of the map have compact type cannot be dropped and obtain a characterization of normed Banach ideal lattices of measurable functions such that the cone of positive functions admits a cone-basis. In Section 12 we extend main results of this paper to certain cases of maps acting on the cone of positive functions from L_1 . In Section 13 we discuss briefly the relationship between additivity and linearity. The rest of the paper is devoted to applications. In fact, in order to understand them only familiarity with the main results of [48] is required; the necessary results are covered in Sections 1, 2 and 4. In Section 14 we show how linear selections of superlinear maps arise in questions related to the existence of a right inverse for a linear operator taking values in L_1 and interpret the main results in this setting. In Section 15 we give a rather superficial treatment to a similar question whether an estimate for a best approximation implies the existence of a linear operator that

realizes it. Finally, in Section 16 we sharpen a result from the theory of analytic functions in the polydisk with the help of linear selections of superlinear maps, which allows us to obtain a weak version of the Corona Theorem on the polydisk in Section 17. The last Section 18 contains some remarks to various parts of the paper.

1. SEMILINEAR SPACES

Let us now introduce the subjects of this paper in a most general form suitable for stating the main results. A set \mathcal{S} equipped with a binary operation of addition $+$ and multiplication by positive real scalars is called a *semilinear space* (over the ordered field of real numbers \mathbb{R}) [22, §1.3] if \mathcal{S} is an Abelian semigroup with respect to addition $+$ and the axioms of a semilinear space $\lambda(x+y) = \lambda x + \lambda y$, $1x = x$, $\lambda(\mu x) = (\lambda\mu)x$, $\lambda x + \mu x = (\lambda + \mu)x$ hold true for all $x, y \in \mathcal{S}$ and $\lambda, \mu > 0$. We say that a semilinear space is *monoid* if it contains an identity $\mathbf{0}$ with respect to addition. Any semilinear non-monoid space \mathcal{S} can be extended to a monoid semilinear space by adding the identity $\mathbf{0}$ and extending the definition of the operations as follows: $x + \mathbf{0} = x$ and $\lambda \mathbf{0} = \mathbf{0}$ for all $x \in \mathcal{S}$ and $\lambda > 0$. Every linear space Y is a monoid semilinear space. Moreover, every convex cone K in a semilinear space \mathcal{S} is a semilinear space. However, not every semilinear space can be identified with a cone in a linear space. For example, the simplest nontrivial Boolean algebra $\{0, 1\}$ with operations $0 + 0 = 0$, $1 + 0 = 0 + 1 = 1 + 1 = 1$, $\lambda 0 = 0$ and $\lambda 1 = 1$ for all $\lambda > 0$ is a two-point semilinear space, but there exists no two-point cone in a linear space (in this paper we work with the linear spaces over the field of reals \mathbb{R} only). In fact, any ring of sets \mathcal{R} is a semilinear space with operations defined by $A + B = A \cup B$ and $\lambda A = A$ for all $A, B \in \mathcal{R}$ and $\lambda > 0$.

For every semilinear space \mathcal{S} the set $\mathcal{C}(\mathcal{S})$ of nonempty convex sets $A, B \subset \mathcal{S}$ equipped with the usual multiplication $\lambda A = \{\lambda x \mid x \in A\}$ and Minkowski sum $A + B = \{x + y \mid x \in A, y \in B\}$ is a semilinear space. It is easy to see that $\mathcal{C}(\mathcal{S})$ is monoid if and only if \mathcal{S} is monoid.

A relation \preccurlyeq on a semilinear space \mathcal{S} is said to be compatible with the semilinear structure of \mathcal{S} if \preccurlyeq is invariant under addition and multiplication by positive constants, that is, $x \preccurlyeq y$ for some $x, y \in \mathcal{S}$ implies $\lambda x \preccurlyeq \lambda y$ and $x + z \preccurlyeq y + z$ for all $\lambda > 0$ and $z \in \mathcal{S}$. There is a natural transitive relation \preccurlyeq associated with every semilinear space \mathcal{S} : $x \preccurlyeq y$ for $x, y \in \mathcal{S}$ if and only if $y - x \in \mathcal{S}$; that is, $x \preccurlyeq y$ means that there exists $z \in \mathcal{S}$ such that $y = x + z$. This relation is compatible with the semilinear structure of \mathcal{S} . We call \preccurlyeq the *intrinsic order* of the semilinear space \mathcal{S} . Relation \preccurlyeq is a partial order if and only if \mathcal{S} is monoid and has no element invertible in $+$ except $\mathbf{0}$; in this case a relationship $x_1 - y_1 = x_2 - y_2 \Leftrightarrow x_1 + y_2 = x_2 + y_1$ on pairs $x - y$ for $x, y \in \mathcal{S}$ is an equivalence relationship, and the set $\mathcal{S} - \mathcal{S} = \{x - y \mid x, y \in \mathcal{S}\}$ with

linear operations induced by \mathcal{S} is a vector space having \mathcal{S} as a generating cone. Note that $\mathcal{C}(\mathcal{S})$ is usually ordered by inclusion \subset , which is a partial order compatible with the semilinear structure of \mathcal{S} . It is easy to see that in general there is no relation between the two orders \preceq and \subset . A semilinear space \mathcal{S} is embedded canonically into $\mathcal{C}(\mathcal{S})$ by the map $x \mapsto \{x\}$.

Suppose that \mathcal{S} and \mathcal{V} are semilinear spaces, \mathcal{V} is equipped with a partial order \leq and $T : K \rightarrow \mathcal{V}$ is a map defined on a convex set $K \subset \mathcal{S}$. We say that T is *superadditive* if K is a cone and $T(x) + T(y) \leq T(x+y)$ for all $x, y \in K$. T is *superlinear* if T is superadditive and positive homogeneous (i. e. $T(\lambda x) = \lambda T x$ for all $x \in K$ and $\lambda > 0$). T is *additive* if K is a cone and $T(x) + T(y) = T(x+y)$ for all $x, y \in K$. T is *linear*¹ if T is additive and positive homogeneous, or, equivalently, if T is affine and positive homogeneous.

A map $S : K \rightarrow \mathcal{V}$ is a *submap* of T if $S \leq T$ pointwise; that is, $S(x) \leq T(x)$ for all $x \in K$. Now suppose that $T : K \rightarrow \mathcal{C}(\mathcal{V})$ is a set-valued map. A map $S : K \rightarrow \mathcal{V}$ is called a *selection* of T if $S(x) \in T(x)$ for all $x \in K$; that is, S is single-valued and $S \leq T$. It is natural to ask whether a given superlinear map has linear selections, and if it does, how the set of all such selections can be parametrized and what additional requirements like continuity or passage through a given point in the graph can be imposed on selections. Let us establish some relevant notions before proceeding to the answers in the following sections.

Suppose that \mathcal{V} is a semilinear space with a partial order \leq compatible with the semilinear structure of \mathcal{V} . Note that even if \mathcal{V} is a monoid we do not require that $0 \leq x$ for all $x \in \mathcal{V}$, and indeed this is not generally the case for $\mathcal{V} = \mathcal{C}(\mathcal{S})$ ordered by inclusion which is our main example. The following two properties are crucial for the main argument of [48]. A set D with a partial order \leq is called *lower directed* if every couple of elements $x, y \in D$ has a lower bound $z \in D$, $z \leq x$, $z \leq y$. We say that the order \leq is *lower complete* in \mathcal{V} if every lower directed set $D \subset \mathcal{V}$ has a greatest lower bound $\inf D \in \mathcal{V}$. We say that a semilinear space \mathcal{V} is *order regular* with respect to a partial order \leq if it is lower complete with respect to \leq and for any lower directed sets $A, B \subset \mathcal{V}$ and $z \in \mathcal{V}$ such that $z \leq x + y$ for all $x \in A$ and $y \in B$ it is also true that $z \leq \inf A + \inf B$. A prominent example of an order regular semilinear space is the space of all nonempty compact convex sets of a locally convex Hausdorff linear topological space ordered by inclusion; we will treat it in Section 4 below (see Proposition 8).

Let \mathcal{S} be a semilinear space equipped with its intrinsic transitive relation \preceq . This relation is extended naturally onto sets $A, B \subset \mathcal{S}$ as

¹*quasilinear* in [48]; this term looks somewhat clumsy, although there probably is a certain point in distinguishing the combination of additivity and positive homogeneity from linearity in some situations.

follows: $A \preceq B$ if and only if $x \preceq y$ for all $x \in A$ and $y \in B$. We use the Minkowski sum $A + B = \{x + y \mid x \in A, y \in B\}$ for $A, B \subset \mathcal{S}$. The initial segments $[0, x]$ are defined by $[0, x] = \{y \in \mathcal{S} \mid y \preceq x\}$ for $x \in \mathcal{S}$. We say that \mathcal{S} has the *interpolation property* if for every finite sets $A, B \subset \mathcal{S}$ such that $A \preceq B$ there exists an intermediate point $x \in \mathcal{S}$ satisfying $A \preceq x \preceq B$; \mathcal{S} has the *Riesz decomposition property* if $[0, x] + [0, y] = [0, x + y]$ for any $x, y \in \mathcal{S}$. A well-known theorem of F. Riesz shows (see e. g. [7, §1.8, Theorem 1.54]; although account in the book is given for ordered vector spaces only, it is easy to verify that the proof remains valid in the semilinear setting) that these two properties are equivalent to one another and equivalent to the following convenient purely algebraic property: for any $z \in \mathcal{S}$ and $\{x_j\}_{j=1}^m, \{y_k\}_{k=1}^n \subset \mathcal{S}$ such that $z = \sum_{j=1}^m x_j = \sum_{k=1}^n y_k$ there exist some $\{z_{jk}\}_{j=1, k=1}^{m, n} \subset \mathcal{S}$ satisfying $x_j = \sum_{k=1}^n z_{jk}$ and $y_k = \sum_{j=1}^m z_{jk}$ for all j and k . We say that a semilinear space \mathcal{S} is a *lattice* or a *Riesz space* if for any $x, y \in \mathcal{S}$ there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$ in \mathcal{S} . It is obvious that every semilinear space which is a lattice satisfies the interpolation property, but the converse is not true (see, e. g., [7, §1.8, Example 1.58]).

We now present a simple but crucial pattern that will be used repeatedly in applications in Sections 14–16 below, mostly with $X = L_1$ and $\mathcal{S} = [L_1]_+ = \{f \in L_1 \mid f \geq 0 \text{ a. e.}\}$. We say that $\|\cdot\|_X$ is a *quasiseminorm* on a semilinear space X if it is a real-valued positive homogeneous function on Z satisfying $\|f + g\|_Z \leq A(\|f\|_Z + \|g\|_Z)$ for all $f, g \in Z$ with a constant A independent of f and g . Surely the norm of a normed space is a quasiseminorm.

Proposition 1 (see [48, Proposition 8]). *Suppose that X is a linear space and $\mathcal{S} \subset X$ is a generating cone for X . Then every linear operator $T : \mathcal{S} \rightarrow Y$ into a linear space Y can be extended to all X by $T(x - y) = Tx - Ty$ for $x, y \in \mathcal{S}$. Suppose also that Y is a normed space, X is a quasiseminormed vector lattice, the quasiseminorm $\|\cdot\|_X$ is additive on the cone $\mathcal{S} = X_+$ of all nonnegative elements of X and T is bounded. Then the extension of T described above is bounded with the same norm.*

Indeed, since T is linear the extension depends only on $x - y$: if $x - y = x_1 - y_1$, then $x + y_1 = x_1 + y$, $T(x) + T(y_1) = T(x_1) + T(y)$, and therefore $T(x) - T(y) = T(x_1) - T(y_1)$. Now suppose that the boundedness assumptions of Proposition 1 are satisfied. Since X is a lattice, the cone X_+ is generating for X , so for arbitrary $z \in X$ we can take $x = z \vee 0$ and $y = z - x$. Then

$$\begin{aligned} \|Tz\|_Y &= \|T(x) - T(y)\|_Y \leq \\ &\|T\|_{X \rightarrow Y} (\|z \vee 0\|_X + \|(z \wedge 0)\|_X) = \\ &\|T\|_{X \rightarrow Y} \|z\|_X = \|T\|_{X \rightarrow Y} \|z\|_X \end{aligned}$$

since the norm $\|\cdot\|_X$ is assumed to be additive on X_+ . The proof of Proposition 1 is complete.

2. ADDITIVE SUBMAPS OF SUPERADDITIVE MAPS

The starting point of the present paper is the following result obtained² by A. Ya. Zaslavskii [48, Theorem 1] that we are going to discuss in this section.

Theorem 2. *Suppose that \mathcal{S} is a semilinear space having the interpolation property, \mathcal{V} is a semilinear space equipped with a partial order \leq compatible with the semilinear structure and \mathcal{V} is order regular with respect to \leq . Then any superadditive map $T : \mathcal{S} \rightarrow \mathcal{V}$ has an additive submap $S \leq T$ given by the formula*

$$(1) \quad S(z) = \inf_{z = \sum_j x_j} \sum_j T(x_j), \quad z \in \mathcal{S},$$

where the infimum is taken over all finite decompositions $z = \sum_j x_j$. Map S is the greatest additive submap of T , i. e. if $R \leq T$ is an additive submap of T then $R \leq S$. If T is positive homogeneous then S is linear.

The proof is straightforward. First, note that for any $z \in \mathcal{S}$ the interpolation property applied to any two decompositions $z = \sum_j x_j$ and $z = \sum_k y_k$ yields a decomposition $z = \sum_{jk} z_{jk}$ satisfying $x_j = \sum_k z_{jk}$ and $y_k = \sum_j z_{jk}$ for all j and k , so by subadditivity

$$\sum_{jk} T(z_{jk}) \leq \sum_j T(x_j) \wedge \sum_k T(y_k),$$

which means that the infimum in (1) exists as an infimum of a lower directed set because of the order regularity of \mathcal{V} with respect to \leq .

Let us now verify that S is additive. Suppose that $x, y \in \mathcal{S}$, $z = x + y$ and $z = \sum_j z_j$ is an arbitrary finite decomposition in \mathcal{S} . The interpolation property of \mathcal{S} in the form of the Riesz decomposition property shows that there exist some $x_j, y_j \in \mathcal{S}$ such that $z_j = x_j + y_j$ for all j . Therefore

$$S(x) + S(y) \leq \sum_j T(x_j) + \sum_j T(y_j) = \sum_j [T(x_j) + T(y_j)] \leq \sum_j T(z_j),$$

and taking infimum over all such decompositions $z = \sum_j z_j$ shows that $S(x) + S(y) \leq S(z) = S(x + y)$. On the other hand, for any decompositions $x = \sum_j x_j$ and $y = \sum_k y_k$ into $x_j, y_k \in \mathcal{S}$ we have $x + y = \sum_j x_j + \sum_k y_k$, so $S(x + y) \leq \sum_j T(x_j) + \sum_k T(y_k)$. This yields $S(x + y) \leq S(x) + S(y)$ by the order regularity of \leq in \mathcal{V} . Thus we have verified the additivity of S .

²Strictly speaking, in [48] it was stated for superlinear maps, but the proof works for additive maps just as well.

The fact that S is the greatest linear submap of T is almost immediate. If R is an additive submap of T then

$$R(x) = \sum_{j=1}^N R(x_j) \leq \sum_{j=1}^N T(x_j)$$

for any $N \in \mathbb{N}$, $x = \sum_{j=1}^N x_j$, $\{x_j\}_{j=1}^N \subset \mathcal{S}$, and it suffices to take the infimum over all such decompositions. Finally, it is easy to see that positive homogeneity of T implies positive homogeneity of S . The proof of Theorem 2 is complete.

Note that under the assumptions of Theorem 2 the additive map S is not necessarily linear if T is merely additive, although some rather weak additional boundedness assumptions imposed on T imply linearity of S in this case; see Section 13. Since the superadditive maps that arise in problems of analysis are usually positive homogeneous, we will not focus our attention on the relationship between additivity and linearity in most of the present work.

An interesting question is, to what extent the Riesz decomposition property and other conditions are actually necessary for the conclusion of Theorem 2? It was mentioned in [48] that there exists an example of cones in \mathbb{R}^n and maps defined on them indicating that the Riesz decomposition property condition on \mathcal{S} cannot be dropped. In fact, Theorem 20 in Section 7 below shows via an application of the Choquet theory and various techniques that for a large class of semilinear spaces that covers the typical applications of the theory the Riesz decomposition property is *necessary* for the conclusion of Theorem 2, yielding at once a wide variety of such examples and also providing an interesting characterization of Choquet simplexes in terms related to the conclusion of Theorem 2. On the other hand, it is also possible to construct a nontrivial example showing that in the present generality the Riesz decomposition property is not *always* necessary for the conclusion of Theorem 2. Let $\mathcal{V} = \{0, 1\}$ be the two-point semilinear space described in Section 1, and define a cone

$$\mathcal{S} = \{(x, y, z) \mid z \geq 0, \text{ and} \\ \text{either } x = 0, 0 \leq y \leq z, \text{ or both } 0 < x < z \text{ and } 0 < y < z\}$$

in \mathbb{R}^3 with open square base³ $0 < x < 1$, $0 < y < 1$ plus one edge $x = 0$, $0 \leq y \leq 1$. It is easy to see that any superlinear map $T : \mathcal{S} \rightarrow \mathcal{V}$ is constant on the topological interior

$$\text{int } \mathcal{S} = \{(x, y, z) \mid z > 0, 0 < x < z, 0 < y < z\},$$

on the rays

$$\mathcal{S}_1 = \{(x, y, z) \mid z > 0, x = y = 0\},$$

³For the definition of the base of a cone see, e. g., in Section 7. It is slightly easier to visualise this example in the affine reformulation; see Section 6.

$$\mathcal{S}_2 = \{(x, y, z) \mid z > 0, x = 0, y = z\}$$

and on the open support cone $\mathcal{S}_0 = \{(x, y, z) \mid z > 0, x = 0, 0 < y < z\}$; furthermore, there are exactly 6 superlinear maps $T : \mathcal{S} \rightarrow \mathcal{V}$ as follows: $T = 0$ everywhere, $T = 1$ on $\text{int } \mathcal{S}$ and $T = 0$ elsewhere, and $T = 1$ on $\mathcal{S}_0 \cup \text{int } \mathcal{S}$ with arbitrary constant values of T on \mathcal{S}_1 and \mathcal{S}_2 . All of these maps are already linear except for the map T defined by $T = 1$ on $\mathcal{S}_0 \cup \text{int } \mathcal{S}$ and $T = 0$ on $\mathcal{S}_1 \cup \mathcal{S}_2$. Expression (1) yields the correct greatest linear submap $S = 1$ on $\text{int } \mathcal{S}$ and $S = 0$ elsewhere, so the conclusion of Theorem 2 still holds true for this map T . However, cone \mathcal{S} does not have the Riesz decomposition property⁴ since, e. g., $(\frac{1}{2}, \frac{1}{2}, 1) \in [(0, 0, 0), (0, \frac{1}{2}, 1) + (\frac{1}{2}, 0, 1)]$ but

$$\left(\frac{1}{2}, \frac{1}{2}, 1\right) \notin \left[(0, 0, 0), \left(0, \frac{1}{2}, 1\right)\right] + \left[(0, 0, 0), \left(\frac{1}{2}, 0, 1\right)\right].$$

It is also easy to construct an example where there is a greatest linear submap for every superlinear map, but it is not always given by the formula (1). For example, take \mathcal{V} as before and define a cone $\mathcal{S} = \{(x, y, z) \mid z > 0, x^2 + y^2 \leq z\}$ with disk $\{z = 1, x^2 + y^2 \leq 1\}$ as a base. In this case linear maps from \mathcal{S} to \mathcal{V} are just constants and maps that are 0 on a single ray intersecting the circle $\{z = 1, x^2 + y^2 = 1\}$ and 1 everywhere else, so there always is a greatest submap for any given superlinear map $T : \mathcal{S} \rightarrow \mathcal{V}$. However, if T is zero on rays intersecting points $z = x = 1, y = \pm 1$ and 1 everywhere else then the greatest linear submap of T is zero, but the expression (1) yields a sublinear map S that takes the value 1 on every point of the circle $\{z = 1, x^2 + y^2 = 1\}$ except $z = x = 1, y = \pm 1$.

As for the space \mathcal{V} , in Proposition 29 of Section 10 we will see that replacing the Riesz decomposition property assumption by a much stronger one (as will be evident in Section 11) of the existence of a cone-basis for \mathcal{S} makes any requirement of Theorem 2 set on the space \mathcal{V} unnecessary. On the other hand, it is fairly easy to construct an example showing that the lower completeness assumption imposed on \mathcal{V} cannot be dropped in general. Take, for example, the space $\mathcal{S} = [L_1([0, 1])]_+ \setminus \{0\}$ of nonnegative summable functions on the unit segment $[0, 1]$ that are not identically 0 (up to a set of measure 0), the space $\mathcal{V} = \mathbb{R}_+ \setminus \{0\}$ of positive real numbers and the map $T : \mathcal{S} \rightarrow \mathcal{V}$ defined by the formula $T(f) = |\{f > 0\}| \int f$; here $|\{f > 0\}|$ denotes the Lebesgue measure of the support of f . Positive homogeneity of T is trivial, and superadditivity is also easy to verify: for any $f, g \in \mathcal{S}$ we

⁴This easily follows from the fact that the closure of the base of \mathcal{S} is not a simplex; see, e. g., [37] or [39] and also Section 7 below.

have

$$\begin{aligned} T(f + g) &= |\{f + g > 0\}| \int f + |\{f + g > 0\}| \int g \geq \\ &|\{f > 0\}| \int f + |\{g > 0\}| \int g = T(f) + T(g). \end{aligned}$$

However, for any $f \in \mathcal{S}$ and $N \in \mathbb{N}$ we can divide the support $\{f > 0\}$ of f into N parts A_j of equal measure and decompose f into a sum of its parts $f_j = f\chi_{A_j}$ on the sets A_j , so for any linear submap R of T we have

$$R(f) = \sum_{j=1}^N R(f_j) \leq \sum_{j=1}^N T(f_j) = \frac{1}{N} |\{f > 0\}| \sum_{j=1}^N \int f_j = \frac{1}{N} T(f)$$

implying that $R(f) < \varepsilon$ for any $\varepsilon > 0$ and therefore $R(f) \notin \mathcal{V}$. Thus T has no linear submaps. Apparently, the nature of restrictions on \mathcal{V} necessary for the conclusion of Theorem 2 to hold depends on the space \mathcal{S} .

3. CERTAIN FACTS ABOUT CONTINUITY

In this section we introduce and discuss the auxiliary facts of topological nature that we need for the main results of this paper; see, e. g., [4] or [32] (and also [46], [45]) for a more detailed exposition of the topological aspects. Let \mathcal{S} and \mathcal{W} be topological spaces. We say that a map $a : \mathcal{S} \rightarrow 2^{\mathcal{W}}$ is *upper semicontinuous* if for every closed set $F \subset \mathcal{W}$ its preimage

$$\begin{aligned} a^{-1}(F) &= \bigcup_{w \in F} a^{-1}(w) = \\ &\bigcup_{w \in F} \{x \in \mathcal{S} \mid a(x) \ni w\} = \{x \in \mathcal{S} \mid a(x) \cap F \neq \emptyset\} \end{aligned}$$

under a is also closed. It is easy to see that a is upper semicontinuous if and only if for every open set $G \subset \mathcal{W}$ the set $\{x \in \mathcal{S} \mid a(x) \subset G\}$ is also open. This implies that for single-valued maps upper semicontinuity is equivalent to the usual continuity.

The following simple proposition, closely following [32, Proposition 4.4, 4.5], shows that, in the case of regular topology, upper semicontinuity is the same as closedness of the graph for maps with closed values. Recall that a topological space \mathcal{W} is called *regular* if every closed set $F \subset \mathcal{W}$ and every point $x \in \mathcal{W} \setminus F$ can be separated from each other by some open neighbourhoods of F and x , i. e. there exists a couple of open sets U and V in \mathcal{W} such that $U \ni x$, $V \supset F$ and $U \cap V = \emptyset$. It is well known that every Hausdorff linear topological space is regular.

Proposition 3. *Suppose that \mathcal{S} and \mathcal{W} are topological spaces, \mathcal{W} is regular and a map $a : \mathcal{S} \rightarrow 2^{\mathcal{W}}$ has nonempty closed values. The following conditions are equivalent.*

- (1) a is upper semicontinuous.
- (2) The graph $\text{gr } a = \{(x, y) \mid x \in \mathcal{S}, y \in a(x)\}$ of the map a is closed in $\mathcal{S} \times \mathcal{W}$.

Indeed, suppose that a is upper semicontinuous under the conditions of Proposition 3. It is sufficient to prove that for every point $(x, z) \notin \text{gr } a$ there exists a neighbourhood of (x, z) that does not intersect $\text{gr } a$. Since $z \notin a(x)$, by regularity of \mathcal{W} there exist open sets $V \ni z$ and $G \supset a(x)$ in \mathcal{W} such that $V \cap G = \emptyset$. By upper semicontinuity of a the set $U = \{x \in \mathcal{S} \mid a(x) \subset G\}$ is open. Then $U \times V$ is a neighbourhood of (x, z) that does not intersect $\text{gr } a$.

Conversely, let the graph $\text{gr } a$ be closed and $F \subset \mathcal{W}$ be an arbitrary closed set; we need to show that $a^{-1}(F) = \{x \in \mathcal{S} \mid a(x) \cap F \neq \emptyset\}$ is also closed. It is easy to see that $a^{-1}(F)$ is the canonical projection of a closed set $(\mathcal{S} \times F) \cap \text{gr } a$ onto \mathcal{S} . Therefore $a^{-1}(F)$ is closed, since the canonical projection from $\mathcal{S} \times \mathcal{W}$ onto \mathcal{S} is an open map. The proof of Proposition 3 is complete.

The following proposition is used in the proof of Theorems 11, 20 and 38 below.

Proposition 4. *Suppose that \mathcal{S} is a semilinear space equipped with a topology such that for every $x \in \mathcal{S}$ and open $U \subset \mathcal{S}$ the set $x + U$ is also open, \mathcal{W} is a Hausdorff linear topological space and \mathcal{V} is the semilinear space of all nonempty compact sets of \mathcal{W} ordered by inclusion. Let $T : \mathcal{S} \rightarrow \mathcal{V}$ be a superlinear map. If T is upper semicontinuous then the maximal linear submap S of T defined by (1) is also upper semicontinuous.*

First of all, it is easy to verify that the space \mathcal{V} is order regular so the infimum in (1) is well-defined; see Proposition 8 in Section 4 below. Let $x \in \mathcal{S}$ and $G \subset \mathcal{W}$ be an open set such that $S(x) \subset G$; it suffices to establish that there exists a neighbourhood $U \subset \mathcal{S}$ of x such that $S(y) \subset G$ for $y \in U$.

Notice that for every lower directed family of compact sets $K_\alpha \subset \mathcal{W}$ such that $\bigcap_\alpha K_\alpha \subset G$ there is an index α such that $K_\alpha \subset G$; otherwise $K_\alpha \setminus G$ would have been a centered family of compact sets and therefore $\bigcap_\alpha K_\alpha \setminus G = \bigcap_\alpha (K_\alpha \setminus G) \neq \emptyset$. Applying this observation to (1), we get a decomposition $x = \sum_{j=1}^n x_j$, $x_j \in \mathcal{S}$, such that $\sum_{j=1}^n T(x_j) \subset G$. Now it is sufficient to find some open sets $G_j \subset \mathcal{W}$ satisfying $T(x_j) \subset G_j$ and $\sum_{j=1}^n G_j \subset G$; once this is done, the sets $U_j = \{z_j \in \mathcal{S} \mid T(z_j) \subset G_j\}$ become open neighbourhoods of x_j by the upper semicontinuity of T , so the set $U = \sum_{j=1}^n U_j$ is an open neighbourhood of x such that $U \subset \{y \in \mathcal{S} \mid S(y) \subset G\}$.

Let $A = \left\{ \{y_j\}_{j=1}^n \mid \sum_{j=1}^n y_j \in G \right\}$. Since \mathcal{W} is a linear topological space, the set $A \subset \mathcal{W}^n$ is open. It contains the direct product $P = \prod_{j=1}^n T(x_j)$ of compact sets which is a compact set in A . The

set $\mathcal{B} = \left\{ \prod_{j=1}^n V \mid V \text{ is a neighbourhood of } 0 \text{ in } \mathcal{W} \right\}$ is a base of the neighbourhoods of 0 for the topology of \mathcal{W}^n , so

$$X = \{y + W \mid y \in P, W \in \mathcal{B}, y + W \subset A\}$$

is an open cover of P . Since P is a compact set, there exists a finite subcover $\left\{ y_k + W_k = \left\{ y_k^{(j)} \right\}_{j=1}^n + \prod_{j=1}^n V_{k,j} \right\}_{k=1}^M \subset X$ of P . Then $G_j = \bigcap \left\{ \bigcup S_j \mid S_j \subset \left\{ y_k^{(j)} + V_{k,j} \right\}_{k=1}^M, T(x_j) \subset \bigcup S_j \right\}$ is an intersection of a finite number of open sets, so G_j is an open set in \mathcal{W} containing $T(x_j)$. On the other hand, $\prod_{j=1}^n G_j \subset \bigcup X \subset A$, so $\sum_{j=1}^n G_j \subset G$ and thus $\{G_j\}_{j=1}^n$ is a suitable collection of open sets. The proof of Proposition 4 is complete.

In order to work with the issues of continuity of linear selections it is desirable to have certain natural agreement between semilinear and topological structures. Let \mathcal{V} be a semilinear space. A set $A \subset \mathcal{V}$ is called *balanced with respect to a point* $x \in A$ if for any $z \in A$ there exists $z' \in A$ such that $x = \frac{1}{2}(z + z')$, i. e. x is a midpoint of a segment $[z, z'] \subset A$ for any $z \in A$. We denote the topological interior of a set A by $\text{int } A$. We say that \mathcal{V} is a *semilinear topological space* if \mathcal{V} is equipped with a topology satisfying the following conditions.

- (1) The semilinear operations of addition and multiplication by positive constants are continuous.
- (2) For every open set $U \subset \mathcal{V}$ and $x \in \mathcal{V}$ the set $x + U$ is also open.
- (3) $\text{int } \mathcal{V} \neq \emptyset$, and for every point $x \in \text{int } \mathcal{V}$ which is not an additive identity and every open set $U \subset \mathcal{V}$ containing x there exists an open set $V \subset U$ which is balanced with respect to x , i. e. for every point $x \in \mathcal{V}$ there exists a base of neighbourhoods of x that are balanced with respect to x .

Naturally, a linear topological space is a semilinear topological space. Throughout the paper, Properties 2 and 3 of the above definition are directly used only in this section in two critical places apiece. Note that if X is a normed lattice of measurable functions then the set X_+ of the functions positive a. e. with the topology induced from X is generally *not* a semilinear topological space as defined above unless $X = L_\infty$ because of property 3. This is of no relevance for the present paper, even though such semilinear spaces X_+ are the focus of it application-wise, because in the case of cones in a normed lattice continuity of the selections usually follows at once from boundedness and linearity. However, to give a more interesting example (which is not used in the present work), there is a natural family of topologies on X_+ turning it into a semilinear topological space. The topologies are defined for every measurable function h satisfying $0 < h \leq 1$ by a base of neighbourhoods for $f \in X$ consisting of the sets $\{g \mid |g - f| < \varepsilon h|f|\}$ for $0 < \varepsilon < 1$,

which for $h = 1$ coincides with the topology induced from X if $X = L_\infty$ and is (much) stronger otherwise.

The following proposition contains an implication from [32, Theorem 4.1].

Proposition 5. *Suppose that \mathcal{S} and \mathcal{W} are semilinear spaces equipped with a topology, and let $T : \mathcal{S} \rightarrow 2^{\mathcal{W}} \setminus \{\emptyset\}$ and $g : \mathcal{W} \rightarrow \mathbb{R}$ be some mappings. Define the support function $g_T^*(x) = \sup_{y \in T(x)} g(y)$ for $x \in \mathcal{S}$. If T and g are upper semicontinuous then g_T^* is also upper semicontinuous.*

Indeed, suppose that $\lambda \in \mathbb{R}$ is an arbitrary number and maps T and g are upper semicontinuous. Then

$$\{x \in \mathcal{S} \mid g_T^*(x) < \lambda\} = \bigcup_{0 < \varepsilon < 1} \{x \in \mathcal{S} \mid T(x) \subset \{y \in \mathcal{W} \mid g(y) < \lambda - \varepsilon\}\}$$

is an open set, so g_T^* is upper semicontinuous.

Observe that under the assumptions of Proposition 5 if the map T and the functional g are both linear then the support function g_T^* is also linear. The following proposition allows us to obtain full continuity of the support function g_T^* from upper semicontinuity of T and g alone.

Proposition 6. *Suppose that \mathcal{S} is a semilinear topological space and $f : \mathcal{S} \rightarrow \mathbb{R}$ is an affine functional on \mathcal{S} . If f is upper (or lower) semicontinuous then f is continuous.*

Indeed, suppose that under the conditions of Proposition 6 linear functional f is upper semicontinuous (the case of lower semicontinuous f is naturally reduced to this case by replacing f with $-f$). This means that for any $x \in \text{int } \mathcal{S}$ which is not an additive identity and $\varepsilon > 0$ there exists an open neighbourhood $U \subset \mathcal{S}$ of x balanced with respect to x such that

$$(2) \quad f(u) \leq f(x) + \varepsilon$$

for any $u \in U$. Since U is balanced with respect to x for any $z \in U$, there exists some $y \in U$ such that $x = \frac{1}{2}(y + z)$, and therefore

$$f(x) = \frac{1}{2}(f(y) + f(z)).$$

Substituting this into (2) with $u = z$ and $u = y$ yields $|f(z) - f(y)| \leq 2\varepsilon$. Thus $|f(z) - f(x)| = \frac{1}{2}|f(z) - f(y)| \leq \varepsilon$ for any $z \in U$. This implies that f is continuous on a set

$$[\text{int } \mathcal{S}]_+ = \{x \in \text{int } \mathcal{S} \mid x \text{ is not an additive identity}\}.$$

Note that $\text{int } \mathcal{S}$ is dense in \mathcal{S} because \mathcal{S} is a convex set. If $[\text{int } \mathcal{S}]_+$ is dense in \mathcal{S} then it follows that f is continuous on the entire semilinear space \mathcal{S} . If, on the other hand, $[\text{int } \mathcal{S}]_+$ is not dense in \mathcal{S} , then there is an additive identity $0 \in \mathcal{S}$ and the set $[\text{int } \mathcal{S}]_+ \cup \{0\}$ is dense in \mathcal{S} . In

this case the set $\{0\}$ is open in \mathcal{S} , so f is automatically continuous at 0 and therefore in all \mathcal{S} . The proof of Proposition 6 is complete.

Proposition 7. *Suppose that \mathcal{S}, \mathcal{W} are semilinear spaces, $g : \mathcal{W} \rightarrow \mathbb{R}$ and $T : \mathcal{S} \rightarrow \mathcal{V} \subset \mathcal{C}(\mathcal{W})$ are linear maps, and g is bounded from above on the sets from \mathcal{V} , i. e. $\sup_{v \in V} g(v) \leq b_V$ with some $b_V \in \mathbb{R}$ for any $V \in \mathcal{V}$. Then the support function g_T^* is linear. If, additionally, \mathcal{S} is a semilinear topological space and \mathcal{W} is equipped with a topology such that both T and g are upper semicontinuous then the support function g_T^* is continuous.*

Linearity of the support function g_T^* is trivial:

$$\begin{aligned} g_T^*(\lambda x + \mu y) &= \sup_{w \in T(\lambda x + \mu y)} g(w) = \\ &= \sup_{u \in T(x), v \in T(y)} g(\lambda u + \mu v) = \sup_{u \in T(x), v \in T(y)} [\lambda g(u) + \mu g(v)] = \\ &= \sup_{u \in T(x)} \lambda g(u) + \sup_{v \in T(y)} \mu g(v) = \lambda g_T^*(x) + \mu g_T^*(y) \end{aligned}$$

for any $x, y \in \mathcal{S}$ and $\lambda, \mu > 0$. The boundedness from above assumed of g on the sets \mathcal{V} guarantees that $g_T^*(x)$ is finite for all $x \in \mathcal{S}$, i. e. g_T^* is a linear functional. Therefore, Propositions 5 and 6 combined yield Proposition 7.

4. LINEAR SELECTIONS OF LINEAR MAPS

In this section we state the main results of this paper concerning linear selections of linear or superlinear maps. A most interesting case to which Theorem 2 can be applied is a semilinear space $\mathcal{V} \subset \mathcal{C}(\mathcal{W})$ of convex sets in some semilinear space \mathcal{W} ordered by inclusion. The semilinear space \mathcal{V} has to be order regular for Theorem 2 to be applicable in this setting. This implies some restrictions on the collection of convex sets \mathcal{V} . The most natural restriction is compactness, but in some situations it is too strong. However, compactness can often be safely replaced by a weaker property. We say that $\mathcal{V} \subset \mathcal{C}(\mathcal{W})$ has *compact type* if for every centered family of sets $A_\alpha \in \mathcal{V}$ their intersection $\bigcap_\alpha A_\alpha$ also belongs to \mathcal{V} . This implies lower completeness of \mathcal{V} , and order regularity is easily verified in this case; see the proof of Proposition 8 below. Theorem 2 then provides for a superlinear map T the greatest linear submap S taking values in \mathcal{V} , so the problem of linear selections of superlinear maps gets reduced to the problem of linear selections of linear maps by Theorem 2.

In order to work with linear selections of linear maps taking values in $\mathcal{V} \subset \mathcal{C}(\mathcal{W})$ we need a dual structure \mathcal{D} of linear functionals on \mathcal{W} satisfying certain properties allowing us to characterize points of sets from \mathcal{V} by values of functionals from \mathcal{D} at these points and vice versa. Suppose that \mathcal{W} is a semilinear space, $\mathcal{D} \subset \mathcal{W}'$ is a set of linear real-valued

functionals on \mathcal{W} and $\mathcal{V} \subset \mathcal{C}(\mathcal{W})$. We denote the affine hyperplanes of a linear functional f on \mathcal{W} by $f^{-1}(c) = \{x \in \mathcal{W} \mid f(x) = c\}$. A linear functional f on \mathcal{D} is said to be *consistent with \mathcal{V}* if for every $A \in \mathcal{V}$ and $c \in \mathbb{R}$ nonemptiness of $f^{-1}(c) \cap A$ implies that $f^{-1}(c) \cap A \in \mathcal{V}$; f is said to be *defining for \mathcal{V}* if $f(A)$ is a segment $[a, b]$ for all $A \in \mathcal{V}$. We say that the set of functionals \mathcal{D} is *exhaustive for \mathcal{W}* if values $\{f(x)\}_{f \in \mathcal{D}}$ uniquely identify every point $x \in \mathcal{W}$.

The main example of a space $\mathcal{V} \subset \mathcal{C}(\mathcal{V})$ good enough for the main results and all but one application presented in this paper is given in the following simple proposition, which outlines the well-known properties of the space of nonempty compact convex sets of a locally convex linear topological space.

Proposition 8. *Suppose that \mathcal{W} is a locally convex Hausdorff linear topological space and a semilinear space $\mathcal{V} = \mathcal{V}(\mathcal{W})$ consists of all nonempty compact convex sets in \mathcal{W} . Then \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(\mathcal{W})$, i. e. \mathcal{V} is closed under the semilinear operations of $\mathcal{C}(\mathcal{W})$. \mathcal{V} has compact type and is order regular. The set \mathcal{W}' of all continuous linear functionals on \mathcal{W} is exhaustive for \mathcal{W} , and all $f \in \mathcal{W}'$ are defining for and consistent with \mathcal{V} .*

Indeed, if $A, B \in \mathcal{V}$ then $A + B$ is convex. If $z_\alpha \in A + B$ is a net then $z_\alpha = x_\alpha + y_\alpha$ for some nets $x_\alpha \in A$ and $y_\alpha \in B$. Since A and B are compact, we can replace the nets by some cofinal ones so that $x_\alpha \rightarrow x$ and $y_\alpha \rightarrow y$ for some $x \in A$ and $y \in B$. This means that $z_\alpha \rightarrow x + y \in A + B$, so $A + B$ is compact and thus closed. We have just shown that $A + B \in \mathcal{V}$ for any $A, B \in \mathcal{V}$. Closedness of \mathcal{V} under multiplication by positive constants is trivial. Therefore, \mathcal{V} is a subspace of $\mathcal{C}(\mathcal{W})$. \mathcal{V} has compact type because intersection of a centered family of nonempty compact convex sets is also a nonempty compact convex set. The inclusion order \subset is lower complete in \mathcal{V} for the same reason, and for any lower directed set $\mathfrak{A} \subset \mathcal{V}$ we have $\inf \mathfrak{A} = \bigcap \mathfrak{A}$. Let us verify the remaining part of the order regularity property for \mathcal{V} . Suppose that $\mathfrak{A}, \mathfrak{B} \subset \mathcal{V}$ are lower directed and $Z \in \mathcal{V}$ satisfies $Z \subset A + B$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. It is evident that for a fixed $B \in \mathcal{V}$ the set $\mathfrak{A} + B = \{A + B \mid A \in \mathfrak{A}\}$ is lower directed, and $\inf(\mathfrak{A} + B) = (\inf \mathfrak{A}) + B$ since $\inf = \bigcap$. Taking infinums over $B \in \mathfrak{B}$ in the same way shows that $Z \subset \inf \mathfrak{A} + \inf \mathfrak{B}$, which proves that \mathcal{V} is order regular. The fact that the set \mathcal{W}' is exhaustive easily follows from a separation theorem. Finally, if $f \in \mathcal{W}'$ is a continuous linear functional then $f(A)$ is compact and convex for any $A \in \mathcal{V}$ as an image of a compact convex set, thus $f(A) = [a, b]$ for some $a, b \in \mathbb{R}$. If $A \in \mathcal{V}$ and $c \in f(A)$ then $f^{-1}(c) \cap A$ is a nonempty convex closed subset of a compact set A , so $f^{-1}(c) \cap A \in \mathcal{V}$. The proof of Proposition 8 is complete.

We remark in passing that in Proposition 8 compact sets can be replaced by nonempty bounded sets that are closed in measure if \mathcal{W} is a Banach ideal lattice of measurable functions on \mathbb{Z} satisfying the Fatou property, as will be discussed in detail in Section 9 below.

Recall that in the present setting Theorem 2 characterizes linear submaps $A : \mathcal{S} \rightarrow \mathcal{V}$ of a superlinear map T under certain assumptions imposed on \mathcal{S} and $\mathcal{V} \subset \mathcal{C}(Y)$ by the explicit formula (1) for the greatest linear submap S of T . The next step in the study of linear selections of T is to describe linear selections $L(A)$ of such a linear submap A . Let $V_A(x) = \{a(x) \mid a \in L(A)\}$ be the set of all possible values of linear selections of A at some point $x \in \mathcal{S}$. In [48, Theorem 2] it was shown (using the Zorn lemma) for the standard case of Y being locally convex and \mathcal{V} consisting of all compact convex sets in Y that the closure of $V_A(x)$ in the weak topology of Y coincides with $A(x)$ and thus the sets $V_A(x)$ are nonempty; consequently, $L(A)$ is also nonempty. While this result provides a positive answer to the question about the existence of linear selections for superlinear maps in this setting, it does not by itself say everything about the set $V_A(x)$, specifically whether $V_A(x) = A(x)$. In other words, is it possible to extend every single-valued linear submap $l \leq A$ defined on a ray spanned by x to a submap of A defined on the entire \mathcal{S} ? Since $V_A(x)$ is convex, it *might* be possible to prove that $V_A(x)$ is closed in the weak topology, which would imply that $V_A(x) = A(x)$. However, there is a more direct approach. Application of a simple technique to the general idea of the proof of [48, Theorem 2] allows us to establish that $V_A(x) = A(x)$ in a more general setting. Moreover, the same technique leads to an *explicit* inductive construction of a linear selection of A passing through a given point in $A(x)$ in addition to the quick proof based on the Zorn lemma, which allows us to dispose of the compact type assumption if the set of defining functionals is finite, which is the case if \mathcal{V} is a finite-dimensional topological space.

Theorem 9. *Suppose that \mathcal{S} and \mathcal{W} are semilinear spaces, \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(\mathcal{W})$, $\mathcal{D} \subset \mathcal{W}'$ is an exhaustive set of linear functionals that are defining for and consistent with \mathcal{V} , and either \mathcal{V} has compact type or \mathcal{D} is finite. Then for every linear map $A : \mathcal{S} \rightarrow \mathcal{V}$, $x \in \mathcal{S}$ and $y \in A(x)$ there exists a linear selection a of A satisfying $a(x) = y$. Suppose, in addition, that \mathcal{S} and \mathcal{W} are equipped with topologies such that all sets in \mathcal{V} are closed, \mathcal{W} is a regular topological space, \mathcal{S} is a semilinear topological space, all functionals in \mathcal{D} are continuous and A is upper semicontinuous. Then there exists a continuous linear selection a of A satisfying $a(x) = y$.*

In Section 5 below we provide a simple proof of Theorem 9 (for the case of \mathcal{V} having compact type) based on the Zorn lemma, which is essentially a refinement of [48, Theorem 2]. We also give a more explicit

construction of the linear selection with required properties (that covers the case of finite \mathcal{D}), which is based on a certain parametrization of points in an arbitrary set $A \in \mathcal{V}$. This construction is based on transfinite recursion in the general case, so it also involves the Zorn lemma indirectly; it is truly explicit only when the set of defining functionals \mathcal{D} for \mathcal{W} is at most countable.

Note that for general spaces \mathcal{S} and \mathcal{V} not every linear set-valued map has a linear selection. Perhaps, the simplest example is the two-point semilinear space $\mathcal{S} = \{0, 1\}$ mentioned before and the positive ray $\mathcal{W} = \{\lambda \mid \lambda > 0\}$. Let $T : \mathcal{S} \rightarrow \mathcal{C}(\mathcal{W})$ be a map defined by $T(f) = \mathcal{W}$ for all $f \in \mathcal{S}$. Map T is linear, but it is easy to see that in this case there are no linear maps $S : \mathcal{S} \rightarrow \mathcal{W}$ at all. Section 11 contains an example that shows that the compact type assumption imposed on \mathcal{V} cannot be dropped from the statement of Theorem 9. It is easy to see, however, that no conditions on \mathcal{V} of this sort are actually necessary if, for example, $\mathcal{S} = \mathbb{R}_+$. More generally, by Proposition 30 in Section 10 below the non-topological conclusion of Theorem 9 is valid if \mathcal{S} has a cone-basis and \mathcal{V} is merely a semilinear subspace of $\mathcal{C}(\mathcal{W})$. However, as it will be evident in Section 11, this approach is rather limited when infinite-dimensional spaces are involved. An interesting particular case of upper semicontinuous maps acting on the space of measures $\mathcal{M}(K)$ with virtually no restrictions placed on \mathcal{V} is treated below in Theorem 23 of Section 8. Another interesting particular case of bounded upper semicontinuous maps acting on the space $[\mathcal{L}_1]_+$ of nonnegative a. e. summable functions on a separable measurable space into convex sets of a Banach ideal space of measurable functions satisfying the Fatou property that are closed in measure is treated in Section 12.

The following result, which is a simple consequence of Theorem 9, is inspired by (and related to) [15, Theorem 2]; see also Theorem 16 in Section 6 below.

Theorem 10. *Suppose that \mathcal{S} and \mathcal{W} are semilinear spaces, \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(\mathcal{W})$, $\mathcal{D} \subset \mathcal{W}'$ is an exhaustive set of linear functionals that are defining for and consistent with \mathcal{V} , and either \mathcal{V} has compact type or \mathcal{D} is finite. Let $T : \mathcal{S} \rightarrow \mathcal{V}$ be a superlinear set-valued map. The following conditions are equivalent.*

- (1) *T is linear.*
- (2) *For any $x \in \mathcal{S}$ and $y \in T(x)$ there exists a linear selection S of T such that $S(x) = y$.*

Indeed, implication $1 \Rightarrow 2$ follows from Theorem 9. Conversely, suppose that condition 2 is satisfied. We need to show that for arbitrary $a, b \in \mathcal{S}$ and $\lambda, \mu > 0$ we have $T(\lambda a + \mu b) = \lambda T(a) + \mu T(b)$. Since T is superlinear, it suffices to establish that $T(\lambda a + \mu b) \subset \lambda T(a) + \mu T(b)$. By condition 2 for $x = \lambda a + \mu b$ and any $y \in T(x)$ there exists a linear

selection S of T satisfying $S(x) = y$. This means that

$$y = S(x) = \lambda S(a) + \mu S(b) \in \lambda T(a) + \mu T(b)$$

as claimed. The proof of Theorem 10 is complete.

Theorems 2 and 9 together yield the following result.

Theorem 11. *Suppose that \mathcal{S} and \mathcal{W} are semilinear spaces, \mathcal{S} has the Riesz decomposition property, \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(\mathcal{W})$ that has compact type and $\mathcal{D} \subset \mathcal{W}'$ is an exhaustive set of linear functionals that are defining for and consistent with \mathcal{V} . Let $T : \mathcal{S} \rightarrow \mathcal{V}$ be a superlinear map. Then T admits a linear selection. Additionally, fix some points $x \in \mathcal{S}$ and $y \in T(x)$. Then the following conditions are equivalent.*

- (1) *There exists a linear selection a of T satisfying $a(x) = y$.*
- (2) *$y \in \sum_{j=1}^N T(x_j)$ for any $N \in \mathbb{N}$ and $\{x_j\}_{j=1}^N \subset \mathcal{S}$ satisfying $x = \sum_{j=1}^N x_j$.*

If, in addition, \mathcal{S} is a semilinear topological space, \mathcal{W} is a locally convex Hausdorff linear topological space, \mathcal{V} consists of all nonempty compact convex sets in \mathcal{W} , all functionals in \mathcal{D} are continuous and T is upper semicontinuous, then the selection in condition 1 can be assumed to be continuous.

The topological clause of Theorem 11 follows from Proposition 4, which shows that under the stated conditions upper semicontinuity of T implies upper semicontinuity of its maximal linear submap determined by the formula (1).

5. TOMOGRAPHICAL COORDINATES

In this section we prove Theorem 9 and discuss some related issues. Suppose that under the conditions of Theorem 9 we are given a linear map $A : \mathcal{S} \rightarrow \mathcal{V}$ and some points $x \in \mathcal{S}$, $y \in A(x)$; we need to construct a suitable linear selection a of A satisfying $a(x) = y$.

First, every $f \in \mathcal{D}$ is bounded on any $K \in \mathcal{V}$, so the *support functions* $R_f(K) = \sup_{z \in K} f(z)$ and $L_f(K) = \inf_{z \in K} f(z)$ are well-defined for all $K \in \mathcal{V}$ and take finite values. Surely, $L_f = -R_{-f}$ and $R_f(K) = f_K^*$, $L_f(K) = -(-f)_K^*$ in terms of functions defined in Proposition 5 above. Let the θ -section of K by f be defined for $0 \leq \theta \leq 1$ by

$$\text{sect}_f(\theta, K) = K \cap f^{-1}((1 - \theta)L_f(K) + \theta R_f(K)).$$

Since f is a defining functional consistent with \mathcal{V} , $\text{sect}_f(\theta, K) \in \mathcal{V}$ for all θ and $\bigcup_{0 \leq \theta \leq 1} \text{sect}_f(\theta, K) = K$; this union is disjoint unless $L_f(K) = R_f(K)$. The following proposition shows that taking such sections of a set from \mathcal{V} is a well-behaved operation.

Proposition 12. *Let \mathcal{S} and \mathcal{W} be semilinear spaces, \mathcal{V} be a semilinear subspace of $\mathcal{C}(\mathcal{W})$, f be a linear functional on \mathcal{V} defining for and*

consistent with \mathcal{V} , and let $T : \mathcal{S} \rightarrow \mathcal{V}$ be a linear map. Then for every fixed $0 \leq \theta \leq 1$ map $\text{sect}_{\theta,f} T$ defined by $\text{sect}_{\theta,f} T(x) = \text{sect}_f(\theta, T(x))$, $x \in \mathcal{S}$, is a linear submap of T . Suppose, additionally, that \mathcal{S} and \mathcal{W} are endowed with topologies so that all sets in \mathcal{V} are closed, \mathcal{W} is a regular topological space, \mathcal{S} is a semilinear topological space, f is continuous and T is upper semicontinuous. Then the map $\text{sect}_{\theta,f} T$ is also upper semicontinuous.

Indeed, $F(x) = ((1 - \theta)L_f(T(x)) + \theta R_f(T(x)))$ defined for $x \in \mathcal{S}$ is a linear functional by Proposition 7. Therefore

$$x \mapsto \text{sect}_{\theta,f} T(x) = \text{sect}_f(\theta, T(x)) = T(x) \cap \{y \in \mathcal{W} \mid f(y) = F(x)\},$$

$x \in \mathcal{S}$, is a linear map, because it is an intersection of two linear maps. Now suppose that the topological assumptions of Proposition 12 are satisfied. Then F is a continuous linear functional by the same Proposition 7, so the graph of the map the map $x \mapsto \{y \in \mathcal{W} \mid f(y) = F(x)\}$ is closed. By Proposition 3 the graph of the map T is also closed, so the graph of the map $\text{sect}_{\theta,f} T$ is closed as an intersection of two closed graphs; in particular, values of $\text{sect}_{\theta,f} T$ are closed. By the same Proposition 3 this means that $\text{sect}_{\theta,f} T$ is upper semicontinuous. The proof of Proposition 12 is complete.

We are now ready to give a simple proof of Theorem 9 for the case of \mathcal{V} having compact type; the other case is covered by a more elaborate construction that will be presented after that. Suppose that the assumptions of Theorem 9 hold true. Let \mathcal{L} be the set of all linear submaps B of A (i. e. $B(z) \subset A(z)$ for all $z \in \mathcal{S}$) satisfying $y \in B(x)$. We induce a partial order \prec in \mathcal{L} by the reverse set inclusion order on the graphs of functions from \mathcal{L} , i. e. for any $D, E \in \mathcal{L}$ we have $D \prec E$ if and only if $\text{gr } E \subset \text{gr } D$. Let \mathfrak{D} be a linearly ordered set $\mathfrak{D} = \{D_\alpha\} \subset \mathcal{L}$. Then the map D defined by $D(x) = \bigcap_\alpha D_\alpha(x)$ for $x \in \mathcal{S}$ also belongs to \mathcal{L} by the compact type of \mathcal{V} , so D is an upper bound for \mathfrak{D} . Thus we can apply the Zorn lemma to obtain a maximal element $M \in \mathcal{L}$. Let us verify that M is single-valued. If, on the contrary, $M(z)$ has at least two points $a, b \in M(z)$ for some $z \in \mathcal{S}$, then by the assumptions there exists a functional $f \in \mathcal{D}$ such that $f(a) \neq f(b)$. Let $\theta = \frac{f(y) - L_f(M(x))}{R_f(M(x)) - L_f(M(x))}$ if $R_f(M(x)) \neq L_f(M(x))$ and $\theta = 0$ otherwise. Then by Proposition 12 map $N = \text{sect}_{\theta,f} M$ also belongs to \mathcal{L} , and since $L_f(N(z)) = R_f(N(z))$, N is a proper submap of M , a contradiction. Thus M is single-valued, and the non-topological claims of Theorem 9 are satisfied. The topological claims are obtained by repeating this argument with \mathcal{L} being the set of all upper semicontinuous linear submaps of A . The simple proof of Theorem 9 is complete.

We now present a construction which leads to another proof of Theorem 9. Suppose that $K \in \mathcal{V}$ and $z \in K$ under the assumptions of

Theorem 9. We want to characterize the position of the point z relative to K in terms of values of functionals \mathcal{D} in a way that would allow us to take advantage of Proposition 12 to build a linear selection of A with graph passing through a prescribed point. In order to do this in the general case we need to invoke transfinite recursion twice, which (at least in its full generality) can be avoided and the construction becomes much more transparent if \mathcal{D} is at most countable. This particular case is also often sufficient for applications, so we are going to treat it first before presenting the construction for the general case (which does not depend on the particular case of at most countable \mathcal{D}).

Let us assume that $\mathcal{D} = \{f_j\}_{j \in I}$, where either $I = \{1 \leq j \leq N\}$ for some $N \in \mathbb{N}$ or $I = \mathbb{N}$. Define the *tomographical coordinates* $\Theta = \{\theta_f\}_{f \in \mathcal{D}} \subset [0, 1]^{\mathcal{D}}$ of a point $z \in K$ relative to $K \in \mathcal{V}$ inductively as follows: $K_0 = K$,

$$\theta_{f_j} = \frac{f_j(z) - L_{f_j}(K_{j-1})}{R_{f_j}(K_{j-1}) - L_{f_j}(K_{j-1})}$$

and $K_j = \text{sect}_{\theta_j, f_j} K_{j-1}$ for $j \in \mathbb{N}$. Here, as usual, $\frac{0}{0} = 0$.

Conversely, since every $f \in \mathcal{D}$ is defining for and consistent with \mathcal{V} and \mathcal{V} has compact type, for every coordinates $\Theta = \{\theta_f\}_{f \in \mathcal{D}} \subset [0, 1]^{\mathcal{D}}$ and $B \in \mathcal{V}$ we can define a sequence of sets $B_j \in \mathcal{V}$ by $B_0 = B$ and $B_j = \text{sect}_{\theta_j, f_j} B_{j-1}$ for $j \in I$. This sequence is nonincreasing, so $C_{B, \Theta} = \bigcap_{j \in I} B_j \in \mathcal{V}$ by the compact type of \mathcal{V} if \mathcal{D} is infinite, otherwise we just have $C_{B, \Theta} = B_N \in \mathcal{V}$. Observe that

$$f_j(C_{B, \Theta}) = (1 - \theta_j)L_{f_j}(B_{j-1}) + \theta_j R_{f_j}(B_{j-1})$$

for all $j \in I$. By assumptions \mathcal{D} is exhaustive for \mathcal{V} , so the set $C_{B, \Theta}$ consists of a single point $B(\Theta) \in B$ uniquely identified by the coordinates Θ . It is easy to see that if $B = K$ then $B_j = K_j$, where K_j are the sets from the construction of the coordinates Θ above. Therefore $K(\Theta) = z$, and thus every point in K can be recovered from its tomographical coordinates.

We now prove Theorem 9 in the case of at most countable \mathcal{D} . Let $\Theta = \{\theta_j\}$ be the tomographical coordinates of y relative to $A(x)$. For any $B \in \mathcal{V}$ the construction above yields a sequence of sets B_j such that $B_0 = B$ and $B_j = \text{sect}_{\theta_j, f_j} B_{j-1}$ for $j \in \mathbb{N}$, and $\{B(\Theta)\} = \bigcap_{j \in I} B_j$. Proposition 12 implies that for all $j \in I$ maps $A_j(z) = [A(z)]_j(\Theta)$, $z \in \mathcal{S}$, are linear, so $z \mapsto [A(z)](\Theta) = \bigcap_{j \in I} [A(z)]_j(\Theta)$ is also a linear map because it is an intersection of linear maps. Therefore $a : \mathcal{S} \rightarrow \mathcal{W}$, $a(z) = \Theta(A(z)) \in A(z)$, is a linear mapping which is a linear selection of A satisfying $a(x) = y$. Now suppose that the topological assumptions of Theorem 9 are satisfied. Proposition 12 together with Proposition 3 implies that all functions A_j have closed graphs, so the graph of a is closed, which by Proposition 3 means that a is continuous. The proof of Theorem 9 for the case of at most countable \mathcal{D} is complete.

Let us now extend the construction and application of tomographical coordinates to the case of arbitrary cardinality of the set \mathcal{D} . In order to do this we have to resort to transfinite recursion; the relevant notions and terminology can be found, for example, in [16, Section 18]. By the Zermelo theorem we can assume that the set \mathcal{D} is well-ordered with respect to an order \prec . Denote by $\mathcal{I}_f = \{g \in \mathcal{D} \mid g \prec f\}$ the initial segment of \mathcal{D} for $f \in \mathcal{D}$.

We are now going to define the coordinates $\Theta = \{\theta_f\}_{f \in \mathcal{D}} \subset [0, 1]^{\mathcal{D}}$ of a point $z \in K$ relative to $K \in \mathcal{V}$. Let $\mathcal{V}_{z,K} = \{A \in \mathcal{V} \mid z \in A \subset K\} \subset \mathcal{V}$, let \mathcal{F}_f be the set of functions $\mathfrak{f} : \mathcal{I}_f \rightarrow \mathcal{V}_{z,K} \times [0, 1]$ with its respective components labelled by $\mathfrak{f} = (\mathfrak{f}_{\mathcal{V}}, \mathfrak{f}_{\theta})$ and let $\mathcal{F} = \bigcup_{f \in \mathcal{D}} \mathcal{F}_f$. We now define a *sequence function* $\mathcal{N} : \mathcal{F} \rightarrow \mathcal{V}_{z,K} \times [0, 1]$ with its respective components labelled by $\mathcal{N} = (\mathcal{N}_{\mathcal{V}}, \mathcal{N}_{\theta})$ as follows. Take $\mathfrak{f} \in \mathcal{F}_f$, $f \in \mathcal{D}$, and define a set $K_{\mathfrak{f}}$ by $K_{\mathfrak{f}} = \bigcap_{g \prec f} \mathfrak{f}_{\mathcal{V}}(g)$ if $\mathcal{I}_f \neq \emptyset$ and by $K_{\mathfrak{f}} = K$ otherwise. The compact type of \mathcal{V} implies that $K_{\mathfrak{f}} \in \mathcal{V}_{z,K}$, so the succeeding coordinate

$$\theta_{\mathfrak{f}} = \frac{f(z) - L_f(K_{\mathfrak{f}})}{R_f(K_{\mathfrak{f}}) - L_f(K_{\mathfrak{f}})}$$

is well-defined. We complete the definition of the sequence function \mathcal{N} by setting $\mathcal{N}(\mathfrak{f}) = (\text{sect}_{\theta,f}(K_{\mathfrak{f}}), \theta_{\mathfrak{f}})$. By the Transfinite Recursion Theorem there exists a unique function $U : \mathcal{D} \rightarrow \mathcal{V}_{z,K} \times [0, 1]$ such that $U(f) = \mathcal{N}(U_f)$ for all $f \in \mathcal{D}$, where U_f is the restriction of U onto the initial segment \mathcal{I}_f . We label the components of U as $U = (U_{\mathcal{V}}, U_{\theta})$ and define the tomographical coordinates $\Theta = \Theta(z, K)$ of the point $z \in K$ relative to K by $\theta_f = U_{\theta}(f)$ for all $f \in \mathcal{D}$.

Now suppose that we are given some coordinates $\Theta = \{\theta_f\}_{f \in \mathcal{D}} \in [0, 1]^{\mathcal{D}}$ and a set $B \in \mathcal{V}$. Let \mathcal{G}_f be the set of functions

$$\mathfrak{g} : \mathcal{I}_f \rightarrow \mathcal{V}_B = \{A \in \mathcal{V} \mid A \subset B\}$$

for all $f \in \mathcal{D}$ and let $\mathcal{G} = \bigcup_{f \in \mathcal{D}} \mathcal{G}_f$. Define a sequence function $\mathcal{M} : \mathcal{G} \rightarrow \mathcal{V}_B$ for $\mathfrak{g} \in \mathcal{G}_f$, $f \in \mathcal{D}$ by $M(\mathfrak{g}) = \text{sect}_{\theta,f} \bigcap_{g \prec f} \mathfrak{g}(g)$ if $\mathcal{I}_f \neq \emptyset$ and $M(\mathfrak{g}) = B$ otherwise. Then by the Transfinite Recursion Theorem there exists a unique function $V : \mathcal{D} \rightarrow \mathcal{V}_B$ satisfying $V(f) = \mathcal{M}(V_f)$ for all $f \in \mathcal{D}$, V_f being the restriction of V onto \mathcal{I}_f . An application of the Transfinite Induction Theorem easily shows that if $f, g \in \mathcal{D}$ and $f \prec g$ then $V(f) \subset V(g)$, which means that $\{V(f)\}_{f \in \mathcal{D}}$ is a centered family of sets in \mathcal{V} , and so $C_{B,\Theta} = \bigcap_{f \in \mathcal{D}} V(f)$ belongs to \mathcal{V} by the compact type of \mathcal{V} . Let us define $B_f = \bigcap_{g \prec f} V(g)$ for $\mathcal{I}_f \neq \emptyset$ and $B_f = B$ otherwise. Surely $f(C_{B,\Theta}) = (1 - \theta_f)L_f(B_f) + \theta_f R_f(B_f)$ for all $f \in \mathcal{D}$. Since \mathcal{D} is exhaustive for \mathcal{V} by the assumptions, the set $C_{B,\Theta}$ consists of a single point $B(\Theta) \in B$ uniquely identified by the coordinates Θ . If under the previous assumptions $B = K$ and $\Theta = \Theta(z, K)$, then another application of the Transfinite Induction Theorem easily shows that $B_f = U_{\mathcal{V}}(f) \ni z$ for all $f \in \mathcal{D}$, so $B(\Theta) = z$, i. e. every point

z in a set $K \in \mathcal{V}$ can be recovered from values of its tomographical coordinates $\Theta(z, K)$.

We now prove Theorem 9 in the general case. Let $\Theta = \{\theta_f\}_{f \in \mathcal{D}}$ be the tomographical coordinates of y relative to $A(x)$. For any $B \in \mathcal{V}$ the above construction yields a collection of sets $B_f(\Theta) = B_f$ such that $B_f = \text{sect}_{\theta_f, f} \bigcap_{g \prec f} B_g$ if $\mathcal{I}_f \neq \emptyset$ and $B_f = B$ otherwise. Then proposition 12 and the Transfinite Induction Theorem together show that $z \mapsto [A(z)]_f(\Theta)$ is a linear map defined for all $z \in \mathcal{S}$, so the map $z \mapsto [A(z)](\Theta) = \bigcap_{f \in \mathbb{D}} [A(z)]_f(\Theta)$ is also linear as an intersection of linear maps, and we also have $[A(x)](\Theta) = y$ by the construction of the tomographical coordinates. Therefore $a : \mathcal{S} \rightarrow \mathcal{W}$, $a(z) = \Theta(A(z)) \in A(z)$, is a linear map which is a linear selection of the map A satisfying $a(x) = y$. Now suppose that the topological assumptions of Theorem 9 are satisfied. Note that $a(z) = \bigcap_{f \in \mathcal{D}} A_f(z)$ for $z \in \mathcal{S}$, where $A_f = \text{sect}_{\theta_f, f} \bigcap_{g \prec f} A_g$ if $\mathcal{I}_f \neq \emptyset$ and $A_f = A$ otherwise. Proposition 12 together with Proposition 3 and the Transfinite Induction Theorem show that all functions A_f have closed graphs, so the graph of the map a is also closed, which by Proposition 3 means that the map a is continuous. The proof of Theorem 9 for the general case is complete.

6. AFFINE SELECTIONS OF CONVEX MAPS

In this section we show how the main results for linear selections of superlinear maps naturally imply corresponding results for affine selections of convex maps (and sometimes vice versa). These results are of independent interest (see, e. g., [40], [41]); we will need them in Section 7 below to obtain a characterization of Choquet simplexes and a partial converse to Theorem 11.

The *suspension* of a convex set $K \subset \mathcal{S}$ in a semilinear space \mathcal{S} is the semilinear space $K_{sus} = \{(\lambda, \lambda x) \mid \lambda > 0, x \in K\} \subset \mathbb{R}_+ \times K$ with respect to the semilinear operations inherited from $\mathbb{R} \times K$. If \mathcal{S} is a linear space then K_{sus} is a cone in the linear space $\mathbb{R} \times \mathcal{S}$. If K is equipped with a topology then there is a natural topology on K_{sus} induced from $\mathbb{R} \times K$. If \mathcal{S} is a semilinear topological space and $\text{int } K \neq \emptyset$ then the natural topology on K_{sus} has a base

$$\mathcal{B}_{(\mu, \mu z)} = \{(\mu - \varepsilon, \mu + \varepsilon) \times (V \cap K) \mid 0 < \varepsilon < \mu,$$

$$V \subset \mathcal{S} \text{ is a balanced open neighbourhood of the point } z\}$$

of open neighbourhoods for every point $(\mu, z) \in K_{sus}$. The base $\mathcal{B}_{(\mu, \mu z)}$ has a sub-base of balanced open neighbourhoods for every point

$$(\mu, z) \in \text{int } K_{sus} = (\text{int } K)_{sus},$$

so K_{sus} is also a semilinear topological space.

Let \mathcal{S} and \mathcal{V} be semilinear spaces, and suppose that \mathcal{V} is equipped with an order \leq consistent with the semilinear structure of \mathcal{V} . Suppose

that $T : K \rightarrow \mathcal{V}$ is a map defined on a convex set $K \subset \mathcal{S}$. The map T is said to be *convex* if

$$(1 - \theta)T(x) + \theta T(y) \leq T((1 - \theta)x + \theta y)$$

for all $x, y \in K$ and $0 < \theta < 1$. T is *affine* if

$$(1 - \theta)T(x) + \theta T(y) = T((1 - \theta)x + \theta y)$$

for all $x, y \in K$ and $0 < \theta < 1$. These properties are related to superlinearity and linearity in the following way (which is well known but somewhat implicit in the literature; see, e. g., [39, Chapter 11]). For every map T on K we define the *natural extension* T_{sus} of T onto K_{sus} by $T((\lambda, \lambda x)) = \lambda T(x)$ for $x \in K$. It is evident that the natural extension T_{sus} is always positive homogeneous and the map T is easily recovered from its natural extension T_{sus} by its restriction $T(x) = T_{sus}((1, x))$ on the set $\{1\} \times K$. It is easy to see that if \mathcal{S} is a semilinear topological space, \mathcal{W} is a semilinear space equipped with a topology in which its semilinear operations are continuous, and T takes values in some subspace $\mathcal{V} \subset \mathcal{C}(\mathcal{W})$ of closed sets, then T is upper semicontinuous if and only if T_{sus} is upper semicontinuous.

Proposition 13. *Suppose that \mathcal{S} and \mathcal{V} are semilinear spaces, $K \subset \mathcal{S}$ is a convex set and $T : K \rightarrow \mathcal{V}$ is a map.*

- (1) *T is affine if and only if T_{sus} is linear (equivalently, additive).*
- (2) *T is convex if and only if T_{sus} is superlinear (equivalently, superadditive).*

Indeed, let $(\lambda, \lambda x)$ and $(\mu, \mu y)$ be arbitrary points in K_{sus} . If T is convex then

$$\begin{aligned} (3) \quad T_{sus}((\lambda, \lambda x)) + T_{sus}((\mu, \mu y)) &= \lambda T(x) + \mu T(y) = \\ &= (\lambda + \mu) \left(\frac{\lambda}{\lambda + \mu} T(x) + \frac{\mu}{\lambda + \mu} T(y) \right) \leq \\ &= (\lambda + \mu) T \left(\frac{\lambda}{\lambda + \mu} x + \frac{\mu}{\lambda + \mu} y \right) = \\ &= T_{sus} \left(\left(\lambda + \mu, (\lambda + \mu) \left(\frac{\lambda}{\lambda + \mu} x + \frac{\mu}{\lambda + \mu} y \right) \right) \right) = \\ &= T_{sus}((\lambda, \lambda x) + (\mu, \mu y)), \end{aligned}$$

so T_{sus} is additive and hence linear. If T is affine then we can replace the inequality sign \leq by the equality sign $=$ in (3) to see that T_{sus} is superadditive and hence superlinear. On the other hand, let $x, y \in K$ be arbitrary points and $0 < \theta < 1$ be an arbitrary number. If T_{sus} is

superadditive then in the same way

$$\begin{aligned}
 (4) \quad (1 - \theta)T(x) + \theta T(y) &= \\
 T_{sus}(((1 - \theta), (1 - \theta)x)) + T_{sus}((\theta, \theta y)) &\leq \\
 T_{sus}(((1 - \theta), (1 - \theta)x) + (\theta, \theta y)) &= \\
 T_{sus}((1, (1 - \theta)x + \theta y)) &= T((1 - \theta)x + \theta y),
 \end{aligned}$$

so T is convex. If T_{sus} is additive then we can replace the inequality sign \leq with the equality sign $=$ in (4) to see that T is affine.

Proposition 13 allows us to transfer directly the results about linear submaps of superlinear maps and their linear selections to the case of maps that are convex and affine. The following is an analogue and consequence of Theorem 2.

Theorem 14. *Suppose that $K \subset \mathcal{S}$ is a convex set in a semilinear space \mathcal{S} , K_{sus} has the Riesz decomposition property, \mathcal{V} is a semilinear space equipped with a partial order \leq compatible with the semilinear structure, and \mathcal{V} is order regular with respect to \leq . Then any convex map $T : K \rightarrow \mathcal{V}$ has an affine submap $S \leq T$ given by the formula*

$$S(x) = \inf_{x = \sum_j \theta_j x_j} \sum_j \theta_j T(x_j), \quad x \in K,$$

where the infimum is taken over all finite convex combinations

$$x = \sum_j \theta_j x_j, \quad 0 \leq \theta_j \leq 1, \quad \sum_j \theta_j = 1.$$

Map S is the greatest affine submap of T , i. e. if $R \leq T$ is another affine submap of T then $R \leq S$.

The following is an analogue and consequence of Theorem 9.

Theorem 15. *Suppose that \mathcal{S} and \mathcal{W} are semilinear spaces, $K \subset \mathcal{S}$ is a convex set, \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(\mathcal{W})$, $\mathcal{D} \subset \mathcal{W}'$ is an exhaustive set of linear functionals that are defining for and consistent with \mathcal{V} , and either \mathcal{D} is finite or \mathcal{V} has compact type. Then for every affine map $A : K \rightarrow \mathcal{V}$, $x \in K$ and $y \in A(x)$ there exists an affine selection a of A such that $a(x) = y$. Suppose, in addition, that \mathcal{S} and \mathcal{W} are equipped with topologies so that all sets in \mathcal{V} are closed, \mathcal{W} is a regular topological space, \mathcal{S} is a semilinear topological space, all functionals in \mathcal{D} are continuous and A is upper semicontinuous. Then there exists a continuous affine selection a of A satisfying $a(x) = y$.*

The following result is an analogue and consequence of Theorem 10.

Theorem 16. *Suppose that \mathcal{S} and \mathcal{W} are semilinear spaces, $K \subset \mathcal{S}$ is a convex set, \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(\mathcal{W})$, $\mathcal{D} \subset \mathcal{W}'$ is an exhaustive set of linear functionals that are defining for and consistent with \mathcal{V} , and either \mathcal{D} is finite or \mathcal{V} has compact type. Let $T : K \rightarrow \mathcal{V}$ be a convex map. The following conditions are equivalent.*

- (1) T is affine.
- (2) For any $x \in \mathcal{S}$ and $y \in T(x)$ there exists an affine selection S of T satisfying $S(x) = y$.

Theorem 16 extends [15, Theorem 2] that was obtained for the finite-dimensional case by a different method involving parametrization of the set of all affine selections via its extreme or exposed points (cf. Theorems 20 and 21 in Section 7 below). Our method seems to be stronger since it does not depend on the existence of convex or exposed points for values \mathcal{V} of the map in question.

The following is an analogue and consequence of Theorem 11.

Theorem 17. *Suppose that \mathcal{S} and \mathcal{W} are semilinear spaces, $K \subset \mathcal{S}$ is a convex set, K_{sus} has the Riesz decomposition property, \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(\mathcal{W})$ that has compact type and $\mathcal{D} \subset \mathcal{W}'$ is an exhaustive set of linear functionals that are defining for and consistent with \mathcal{V} . Let $T : K \rightarrow \mathcal{V}$ be a convex map. Then T admits an affine selection. Additionally, fix some points $x \in K$ and $y \in T(x)$. Then the following conditions are equivalent.*

- (1) *There exists an affine selection a of T such that $a(x) = y$.*
- (2) *$y \in \sum_{j=1}^N \theta_j T(x_j)$ for any $N \in \mathbb{N}$, $\{\theta_j\}_{j=1}^N \subset [0, 1]$ and a sequence $\{x_j\}_{j=1}^N \subset K$ such that $x = \sum_{j=1}^N \theta_j x_j$ and $\sum_{j=1}^N \theta_j = 1$.*

If, additionally, \mathcal{S} is a semilinear topological space, \mathcal{W} is a locally convex Hausdorff linear topological space, \mathcal{V} consists of all nonempty compact convex sets in \mathcal{W} , all functionals in \mathcal{D} are continuous and T is upper semicontinuous, then the selection in condition 1 can be taken continuous.

7. CHARACTERIZATION OF CHOQUET SIMPLEXES

In this section we will demonstrate that under suitable restrictions on the semilinear spaces the existence of linear selections for arbitrary superlinear maps implies that the semilinear space the maps are acting on is a lattice and thus has the Riesz decomposition property, thus establishing a partial converse to Theorem 11. This result relies heavily on the Choquet theory; specifically, with the help of the Choquet-Meyer theorem we show that under some appropriate restrictions on the spaces every convex map $T : K \rightarrow \mathcal{V} \subset \mathcal{C}(\mathcal{W})$ admits an affine selection if and only if K is a Choquet simplex.

First, we need to introduce a few relevant notions. The suspension construction introduced in Section 6 above relates a convex semilinear space to every convex set of a semilinear space in a particular way. It is well known that many (but not all) semilinear spaces can be represented as suspensions of convex sets. Let \mathcal{S} be a semilinear space. We say that a convex set $B \subset \mathcal{S}$ is a *base* of \mathcal{S} if for every $y \in \mathcal{S}$ which is not an additive identity there exist unique $\lambda > 0$ and $x \in B$ such that $y = \lambda x$.

It is easy to see that for every convex set $K \subset \mathcal{V}$ in a semilinear space \mathcal{V} the set $\{1\} \times K$ is a base for the suspension K_{sus} .

It is said that convex sets $B \subset \mathcal{S}$ and $K \subset \mathcal{V}$ in semilinear spaces \mathcal{S} and \mathcal{V} are *affinely isomorphic* if there exists an affine bijection $T : B \rightarrow K$ onto K ; the inverse map T^{-1} is then also an affine bijection $T : K \rightarrow B$ onto B . For example, the set $\{1\} \times K \subset K_{sus}$ is affinely isomorphic to K . It is easy to see that (see [39, Proposition 11.2]) an affine isomorphism R between bases $B \subset \mathcal{S}$ and $K \subset \mathcal{V}$ of respective semilinear spaces \mathcal{S} and \mathcal{V} can be extended to a linear isomorphism T between \mathcal{S} and \mathcal{V} in the following way: for $y \in \mathcal{S}$ we take $\lambda > 0$ and $x \in B$ such that $y = \lambda x$ and set $T(y) = \lambda R(x)$; moreover, if semilinear spaces \mathcal{S} and \mathcal{V} are equipped with topologies such that their respective semilinear operations are continuous and R is continuous then T is also continuous; thus if R is a homeomorphism then T is an affine homeomorphism between \mathcal{S} and \mathcal{V} . This implies that if a semilinear space \mathcal{V} has a base K then \mathcal{V} is affinely isomorphic to K_{sus} , and if \mathcal{V} is a semilinear topological space then this isomorphism is also a homeomorphism. In other words, any semilinear space \mathcal{V} with a base K is just the suspension K_{sus} of the base K up to an affine isomorphism. Not every cone has a cone-base, but it is a fairly common property. Let us mention a few examples. The cone of finite nonnegative measures $\mathcal{M}_+(\Omega)$ on a measurable space Ω has the set of probability measures $\mathcal{M}_{+,1}$ as a base. Likewise, we can restrict measures in this example to those that are absolutely continuous with respect to a fixed measure μ ; this leads to the fact that the cone of all nonnegative μ -summable functions $[L_1(\Omega, \mu)]_+$ has a base, for example the set of measurable functions $f \geq 0$ such that $\int f d\mu = 1$. On the other hand, if a cone C is not generating (i. e. if $C \cap -C$ has more than one point) then C does not have a base. The cone $K = \{(x, y) \mid (y = 0 \wedge x \geq 0) \vee y > 0\}$ corresponding to the lexicographic ordering of the plane \mathbb{R}^2 is generating, but it is easy to see that it does not have a base. For more information on cones that admit bases see, e. g., [17, 1.9].

We now introduce the last set of relevant notions before stating the main result of this section; see, e. g., [39, Chapter 9–11] for a detailed introduction into Choquet theory. Let K be a compact convex set in a locally convex Hausdorff linear topological space \mathcal{V} . Then for every probability Baire measure μ on K by [39, Theorem 9.1] there exists a unique point $x \in K$ such that for all continuous affine functionals ℓ on K one has $\ell(x) = \int_K \ell(y) d\mu(y)$. The point x is called a *barycenter* or *the center of mass* of the measure μ . If \mathcal{V} is finite-dimensional then this is equivalent to $x = \int_K y d\mu(y)$. Let $\mathcal{E}(K)$ be the set of extreme points of K . The famous Choquet theorem [39, Theorem 10.7] states that if K is a *metrizable* compact convex set in a locally convex Hausdorff linear topological space then the set of extreme points $\mathcal{E}(K)$ is Baire measurable and for any $x \in K$ there exists a probability Baire measure

μ on K such that $\mu(\mathcal{E}(K)) = 1$ and x is the barycenter of μ . A convex set $K \subset \mathcal{S}$ in a semilinear space \mathcal{S} is called an *algebraic simplex* or just *simplex* if K_{sus} is a lattice. If, additionally, \mathcal{S} is a locally convex Hausdorff linear topological space and K is a compact set then K is called a *Choquet simplex*. The well-known corollary to the Choquet-Meyer theorem (that we will introduce in a moment) [39, Theorem 11.12] states the following: if K is a metrizable compact set in a locally convex Hausdorff linear topological space then K is a Choquet simplex if and only if for any $x \in K$ there exists a *unique* probability measure μ_x on K with barycenter x such that $\mu_x(\mathcal{E}(K)) = 1$.

But what if K is not metrizable? This case is also fairly well understood. Without the assumption of metrizability the set of extreme points $\mathcal{E}(K)$ is not necessarily Baire measurable, so we need more subtle terms to express the condition that a measure μ is located at $\mathcal{E}(K)$. We denote by $C(B)$ the space of all continuous (real-valued) functions on a set $B \subset K$, by $\mathcal{A}(K) \subset C(K)$ the space of all continuous affine functions on K and by $\mathcal{C}(K) \subset C(K)$ the set of all continuous convex functions on K . For a function $f \in \mathcal{C}(K)$ its *concave envelope* \hat{f} is defined by

$$\hat{f}(x) = \inf \{g(x) \mid -g \in \mathcal{C}(K), g \geq f\}, x \in K.$$

The set $\mathcal{C}(K)$ can be replaced by the set of continuous affine functions $\mathcal{A}(K)$ in this definition.

Choquet-Meyer Theorem ([39, Theorem 11.5]⁵). *Let K be a compact convex set in a locally convex Hausdorff linear topological space. Recall that $\mathcal{M}_{+,1}(K)$ is the space of Baire probability measures on K . The following conditions are equivalent.*

- (1) *K is a simplex.*
- (2) *For each $f \in \mathcal{C}(K)$ its concave envelope \hat{f} is affine.*

Additionally, if K is metrizable then there are two more equivalent conditions.

- (3) *For any $f \in \mathcal{C}(K)$ and $\mu \in \mathcal{M}_{+,1}(K)$ having $x \in K$ as the barycenter it is true that $\int_K f d\mu = \hat{f}(x)$.*
- (4) *For each $x \in K$ there exists a unique Baire probability measure $\mu \in \mathcal{M}_{+,1}(K)$ having x as its barycenter.*

We are now ready to state the main result.

Theorem 18. *Suppose that \mathcal{S} is a semilinear topological space with a base K which is a compact convex set in a locally convex Hausdorff linear topological space. Let \mathcal{W} be any nontrivial locally convex Hausdorff*

⁵ We state the two last equivalent conditions for the case of metrizable K only to avoid discussing maximal measures in the Choquet order since we do need it in the present work.

linear topological space and $\mathcal{V}(\mathcal{W}) \subset \mathcal{C}(\mathcal{W})$ be the space of all nonempty compact convex sets in \mathcal{W} . The following conditions are equivalent.

- (1) K is a Choquet simplex.
- (2) \mathcal{S} is a lattice.
- (3) \mathcal{S} has the Riesz decomposition property.
- (4) For any superlinear map $T : \mathcal{S} \rightarrow \mathcal{V}(\mathcal{W})$ there exists a linear selection S of T .
- (5) For any convex map $T : K \rightarrow \mathcal{V}(\mathcal{W})$ there exists an affine selection S of T .
- (6) Every superlinear map $T : \mathcal{S} \rightarrow \mathcal{V}(\mathbb{R})$ has a linear selection S .
- (7) Every convex map $T : K \rightarrow \mathcal{V}(\mathbb{R})$ has an affine selection S .

Note that equivalence of conditions 1–3 of Theorem 18 (and also condition 8 of Theorem 20 below) is rather well known; see, e. g., [37, §3]; however, we establish it using nothing more than the Choquet-Meyer theorem and the results of this paper. All sequential transitions $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Leftrightarrow 5 \Rightarrow 6 \Leftrightarrow 7$ are either trivial or have already been covered: $1 \Rightarrow 2$ is the definition of a Choquet simplex which is that K_{sus} is a lattice together with the fact that K_{sus} is affinely isomorphic to \mathcal{S} because K is a base of \mathcal{S} , $2 \Rightarrow 3$ is a simple property mentioned in the introduction, $3 \Rightarrow 4$ is a consequence of Theorem 11, $4 \Leftrightarrow 5$ and $6 \Leftrightarrow 7$ follow from Proposition 13, and implication $5 \Rightarrow 7$ (or $4 \Rightarrow 6$) is trivial as we can immerse $\mathcal{V}(\mathbb{R})$ into $\mathcal{V}(\mathcal{W})$ by mapping \mathbb{R} onto a line in \mathcal{W} . The remaining implication $7 \Rightarrow 1$ is a simple consequence of the Choquet-Meyer theorem. Indeed, suppose that $f \in \mathcal{C}(K)$; we need to show that its concave envelope \hat{f} is affine. Define a set-valued map $T : K \rightarrow \mathcal{V}(\mathbb{R})$ by $T(x) = \{y \mid f(x) \leq y \leq \hat{f}(x)\}$, $x \in K$. It is easy to see that T is a convex map because it is an intersection of two convex maps $\{y \geq f(x)\}$ and $\{y \leq \hat{f}(x)\}$. Condition 7 implies that there exists an affine selection S of T . Then $S \geq \hat{f}$ by the definition of concave envelope and $S \leq \hat{f}$ because S is a selection of T , thus $S = \hat{f}$ and \hat{f} is affine. The proof of Theorem 18 is complete.

If K is a Choquet simplex and $x \in K$, let us denote by μ_x the unique Baire probability measure on K such that $\mu_x(\mathcal{E}(K)) = 1$ and μ_x has barycenter x . When is the map $x \mapsto \mu_x$ continuous? The answer is provided by the following particular case of a well known result.

Theorem 19 ([39, Theorem 11.13]⁶). *Let K be a metrizable Choquet simplex in a locally convex Hausdorff linear topological space. The following conditions are equivalent.*

- (1) The set $\mathcal{E}(K)$ of extreme points of K is closed.

⁶Again, we avoid discussing maximal measures in the Choquet order by making use of the fact that the set of maximal measures coincides with the family of measures $\{\mu_x\}_{x \in K}$ if the compact convex set K is metrizable.

- (2) The map $x \mapsto \mu_x$ is continuous in the weak* topology $\sigma(\mathcal{M}(K), C(K))$ of $\mathcal{M}(K)$.
- (3) Family $\{\mu_x\}_{x \in K}$ is closed in the weak* topology of $\mathcal{M}(K)$.
- (4) The convex envelope \hat{f} is continuous for all $f \in \mathcal{C}(A)$.

If any (and hence all) of the conditions of Theorem 19 is satisfied, K is called a *Bauer simplex*. In this case we can refine the characterization provided by Theorem 18 as follows.

Theorem 20. *Suppose that \mathcal{S} is a semilinear topological space with a base K which is a compact metrizable convex set in a locally convex Hausdorff linear topological space and the set $\mathcal{E}(K)$ of extreme points of K is closed. Let \mathcal{W} be any nontrivial locally convex Hausdorff linear topological space and $\mathcal{V}(\mathcal{W}) \subset \mathcal{C}(\mathcal{W})$ be the space of all nonempty compact convex sets in \mathcal{W} . Then the following conditions are equivalent.*

- (1) K is a Choquet (and thus Bauer) simplex.
- (2) \mathcal{S} is a lattice.
- (3) \mathcal{S} has the Riesz decomposition property.
- (4) For any upper semicontinuous superlinear map $T : \mathcal{S} \rightarrow \mathcal{V}(\mathcal{W})$ there exists a continuous linear selection S of T .
- (5) For any upper semicontinuous convex map $T : K \rightarrow \mathcal{V}(\mathcal{W})$ there exists a continuous affine selection S of T .
- (6) Any upper semicontinuous superlinear map $T : \mathcal{S} \rightarrow \mathcal{V}(\mathbb{R})$ has a continuous linear selection S .
- (7) Any upper semicontinuous convex map $T : K \rightarrow \mathcal{V}(\mathbb{R})$ has a continuous affine selection S .
- (8) $\mathcal{A}(K) = C(\mathcal{E}(K))$ in the sense that the restriction of all functions from $\mathcal{A}(K)$ onto the set $\mathcal{E}(K)$ of extreme points of K coincides with $C(\mathcal{E}(K))$.

All sequential implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Leftrightarrow 5 \Rightarrow 6 \Leftrightarrow 7$ are established as in Theorem 18, making use of the topological clauses of Theorem 11. Implication $8 \Rightarrow 1$ follows from the particular case of the Choquet-Meyer theorem (see also [39, Theorem 11.5]). If condition 8 is satisfied and there are two Baire measures μ_1 and μ_2 concentrated on $\mathcal{E}(K)$ having $x \in K$ as the barycenter then their difference $\mu = \mu_1 - \mu_2$ satisfies $\int_K \ell(y) d\mu(y) = 0$ for all $\ell \in \mathcal{A}(K) = C(\mathcal{E}(K))$ which is a pre-dual for the space of Borel measures $\mathcal{M}(\mathcal{E}(K))$ on $\mathcal{E}(K)$ (note that the σ -algebras of Borel and Baire sets on K coincide because K is metrizable) having finite variation, and therefore $\mu = 0$, which proves the uniqueness condition in the Choquet-Meyer theorem. The remaining implication $7 \Rightarrow 8$ is not quite as trivial. Let us prove it.

First observe that a map $M : K \rightarrow \mathcal{M}(\mathcal{E}(K))$ defined by

$$M(x) = \{\mu \mid \mu \text{ is a Baire measure on } K, \\ \mu(\mathcal{E}(K)) = 1 \text{ and } x \text{ is the barycenter of } \mu\}$$

is convex. Indeed, for any $0 < \theta < 1$, $x, y \in K$, $\mu \in M(x)$, $\nu \in M(y)$ and a continuous affine functional ℓ on K we have

$$\begin{aligned} \ell((1 - \theta)x + \theta y) &= (1 - \theta)\ell(x) + \theta\ell(y) = \\ &= (1 - \theta) \int_K \ell(z) d\mu(z) + \theta \int_K \ell(z) d\nu(z) = \\ &= \int_K \ell(z) d[(1 - \theta)\mu + \theta\nu](z), \end{aligned}$$

so $(1 - \theta)\mu + \theta\nu \in M((1 - \theta)x + \theta y)$ and thus

$$(1 - \theta)M(x) + \theta M(y) \subset M((1 - \theta)x + \theta y).$$

Now fix $f \in C(\mathcal{E}(K))$; we need to find a function S in $\mathcal{A}(K)$ that coincides with f on $\mathcal{E}(K)$. Define a map $T : K \rightarrow \mathcal{V}(\mathbb{R})$ by

$$T(x) = (M \circ f)(x) = \left\{ \int_{\mathcal{E}(K)} f(z) d\mu(z) \mid \mu \in M(x) \right\}$$

for $x \in K$, which is just composition of f and M . It is easy to derive from convexity of M that T is a convex map. Let us verify that the graph of T is closed. Suppose that $\{x_n\} \subset K$ is an arbitrary sequence converging to some $x \in K$ and a sequence $y_n \in T(x_n)$ converges to some $y \in \mathbb{R}$; we need to show that $y \in T(x)$. Note that $y_n = \int_{\mathcal{E}(K)} f(z) d\mu_n(z)$ for some measures $\mu_n \in M(x_n)$. Observe that since $\mathcal{E}(K)$ is closed in K , $\mathcal{E}(K)$ is a compact set, and thus $C(\mathcal{E}(K))$ is a Banach space and its dual is $\mathcal{M}(\mathcal{E}(K))$. The sequence $\{\mu_n\}$ is bounded in $\mathcal{M}(\mathcal{E}(K))$, so we can pass to a subsequence and assume that μ_n converges in the weak* topology to some $\mu \in \mathcal{M}(\mathcal{E}(K))$. As for all $\ell \in \mathcal{A}(K)$ (and, in particular, for $\ell = 1$) we have $\ell(x_n) = \int_{\mathcal{E}(K)} \ell(z) d\mu_n(z)$, passage to the limit shows that $\mu \in M(x)$. Surely y_n converges to $y = \int_{\mathcal{E}(K)} f(z) d\mu(z) \in \Phi(x)$, so the graph of T is indeed closed and thus T is upper semicontinuous by Proposition 3.

If condition 7 of Theorem 20 is satisfied, then T has a continuous affine selection S . Note that $M(x) = \{\delta_x\}$ for $x \in \mathcal{E}(K)$ where δ_x is a unit point mass concentrated at x ; this follows at once from the Bauer theorem (see, e. g., [39, Theorem 9.3]). Therefore $T(x) = \{f(x)\}$ and $S(x) = f(x)$ for $x \in \mathcal{E}(K)$, i. e. S coincides with f on the set of extreme points $\mathcal{E}(K)$. We have just verified implication $7 \Rightarrow 8$. The proof of Theorem 20 is complete.

We mention that in the finite-dimensional case, i. e. when $\mathcal{S} \subset \mathbb{R}^{n+1}$ or $K \subset \mathbb{R}^n$, Theorems 18 and 20 can be obtained with much simpler means by making use of cone-bases⁷ and simple arithmetic instead of tomographical coordinates and the Riesz decomposition property; see the end of Section 10.

⁷Plural of “cone-basis”.

8. PARAMETRIZATION OF THE SET OF SELECTIONS

In this section we explore how the results of the previous Section 7 can be used in certain cases to naturally parametrize the set of all affine selections of a given convex map by their values on the extreme points of the simplex, which leads to some extensions of Theorems 17 and 11. These extensions, however, are applicable essentially only to the case of upper semicontinuous maps acting on the semilinear space of Borel measures $\mathcal{S} = \mathcal{M}_+(K)$ with the weak* topology on a metrizable compact set K .

Condition 8 of Theorem 20 in Section 7 suggests a natural parametrization for the set of continuous affine selections of an upper semicontinuous convex set-valued map into $\mathcal{V}(\mathbb{R})$ by its values on $\mathcal{E}(K)$. More generally, an easy application of Choquet theory yields a parametrization for all continuous affine selections of an upper semicontinuous convex map acting into a Banach space X by its values on the set $\mathcal{E}(K)$. In order to do this we employ the Bochner integral instead of the scalar one; see, e. g., [47, §4, §5, Chapter V]. Note that if $A \subset K$ is a Baire measurable subset of K and $f : A \rightarrow X$ is a continuous function then $f(A)$ is separable in X because A is separable (the closure of A in K is a metrizable compact set that admits ε -nets for any $\varepsilon > 0$); f is weakly measurable because f is weakly continuous (i. e. with respect to the weak topology $\sigma(X, X^*)$ in X), and thus by the Pettis theorem f is strongly measurable. Therefore the Bochner integral $\int_A f d\mu$ is well-defined for any σ -finite Baire measure μ on K if $\int_A \|f\|_X d\mu < \infty$; in particular, this is the case if $\mu(K) = 1$ and A is closed and thus f is bounded on A . We denote by $C(A \rightarrow X)$ the set of X -valued continuous functions on A .

Theorem 21. *Suppose that K is a metrizable compact set in a locally convex Hausdorff linear topological space, K is a Bauer simplex (i. e. the set $\mathcal{E}(K)$ of extreme points of K is closed in K), μ_x is the measure on $\mathcal{E}(K)$ with barycenter x for $x \in K$, X is a Banach space, $T : K \rightarrow 2^X \setminus \{\emptyset\}$ is an upper semicontinuous convex set-valued map and values of T are closed in X . Then for every $f \in C(\mathcal{E}(K) \rightarrow X)$ satisfying $f(x) \in T(x)$ for all $x \in \mathcal{E}(K)$ function $S_f(x) = \int_{\mathcal{E}(K)} f(y) d\mu_x(y)$, $x \in K$, is a continuous affine selection S of T . Conversely, any continuous affine selection S of T has the representation $S = S_f$ where f is the restriction of S onto $\mathcal{E}(K)$.*

Indeed, by the proof of Theorem 20 the map $x \mapsto \mu_x$ is affine, and therefore S_f is also affine. Let us verify that S_f is continuous. Suppose that $x_n \in K$ is an arbitrary sequence converging to some $x \in K$ in K ; we need to establish that $S_f(x_n) \rightarrow S_f(x)$. Let $\varepsilon_k > 0$ be a sequence of numbers converging to 0. Since $f(\mathcal{E}(K))$ is a compact set, for every k there exists a finite cover $\mathcal{B}_k = \{B_j^k\}_{j=1}^{m_k}$ of $f(\mathcal{E}(K))$ by balls B_j^k with

respective centers z_j^k having radius ε_k . Let $\{\varphi_j^k\}_{j=1}^{m_k}$ be a continuous partition of unity on $f(\mathcal{E}(K))$ subordinate to \mathcal{B}_k . We introduce approximating functions $f_k(y) = \sum_{j=1}^{m_k} \varphi_j^k(f(y))z_j^k$ for $y \in \mathcal{E}(K)$; let us verify that the sequence f_k converges to f uniformly in $\mathcal{E}(K)$. Suppose that $y \in K$. Then $f(y) \in B_j^k$ for some j , condition $\varphi_l^k(f(y)) > 0$ implies that balls B_j^k and B_l^k intersect, and thus $\|z_l^k - z_j^k\|_X \leq 2\varepsilon_k$ because both of these balls have radius ε_k . Therefore

$$\begin{aligned}
 (5) \quad \|f_k(y) - f(y)\|_X &= \left\| \sum_{\{l|\varphi_l^k(f(y))>0\}} \varphi_l^k(f(y))z_l^k - z_j^k + z_j^k - f(y) \right\|_X \leq \\
 &\left\| \sum_{\{l|\varphi_l^k(f(y))>0\}} \varphi_l^k(f(y))z_l^k - z_j^k \right\|_X + \|z_j^k - f(y)\|_X \leq \\
 &2\varepsilon_k + \varepsilon_k = 3\varepsilon_k.
 \end{aligned}$$

Since K is a Bauer simplex, the map $x \mapsto \mu_x$ is continuous with respect to the weak* topology of $\mathcal{M}(\mathcal{E}(K))$. This means that for any k there exists some N_k such that for all $n \geq N_k$ we have

$$(6) \quad \left| \int_{\mathcal{E}(K)} \varphi_j^k(f(y))d\mu_{x_n}(y) - \int_{\mathcal{E}(K)} \varphi_j^k(f(y))d\mu_x(y) \right| \leq \frac{\varepsilon_k}{m_k \left(1 + \bigvee_{j=1}^{m_k} \|z_j^k\|_X\right)}$$

for all j , $1 \leq j \leq m_k$. Combining (6) and (5) together and taking advantage of the properties of the partition of unity, we easily arrive at

$$\begin{aligned}
 &\|S_f(x_n) - S_f(x)\|_X \leq \\
 &\|S_f(x_n) - S_{f_k}(x_n)\|_X + \|S_{f_k}(x_n) - S_{f_k}(x)\|_X + \|S_{f_k}(x) - S_f(x)\|_X \leq \\
 &\left\| \int_{\mathcal{E}(K)} [f(y) - f_k(y)]d\mu_{x_n}(y) \right\|_X + \\
 &\left\| \int_{\mathcal{E}(K)} \left[\sum_{j=1}^{m_k} \varphi_j^k(f(y))z_j^k \right] d\mu_{x_n}(y) - \int_{\mathcal{E}(K)} \left[\sum_{j=1}^{m_k} \varphi_j^k(f(y))z_j^k \right] d\mu_x(y) \right\|_X +
 \end{aligned}$$

$$\begin{aligned}
& \left\| \int_{\mathcal{E}(K)} [f_k(y) - f(y)] d\mu_x(y) \right\|_X \leq \\
& \int_{\mathcal{E}(K)} \|f(y) - f_k(y)\|_X d\mu_{x_n}(y) + \\
& \sum_{j=1}^{m_k} \left| \int_{\mathcal{E}(K)} \varphi_j^k(f(y)) d\mu_{x_n}(y) - \int_{\mathcal{E}(K)} \varphi_j^k(f(y)) d\mu_x(y) \right| \|z_j^k\|_X + \\
& \int_{\mathcal{E}(K)} \|f_k(y) - f(y)\|_X d\mu_x(y) \leq 3\varepsilon_k + \varepsilon_k + 3\varepsilon_k = 7\varepsilon_k
\end{aligned}$$

for all $n \geq N_k$, which means that $S_f(x_n)$ converges to $S_f(x)$ as claimed. It is easy to verify that S_f is a selection of T . Indeed, for any convex combination $x = \sum_{j=1}^N \theta_j x_j$, $0 \leq \theta_j \leq 1$, $\sum_{j=1}^N \theta_j = 1$ of extreme points $x_j \in \mathcal{E}(K)$ we have $\mu_x = \sum_{j=1}^N \theta_j \delta_{x_j}$ where δ_{x_j} is a point mass concentrated at x_j . Therefore

$$S_f(x) = \sum_{j=1}^N \theta_j f(x_j) \in \sum_{j=1}^N \theta_j T(x_j) \subset T\left(\sum_{j=1}^N \theta_j x_j\right) = T(x)$$

by the convexity of T , so $S_f \in T$ on the convex hull $\text{co } \mathcal{E}(K)$ of $\mathcal{E}(K)$. Since T is upper semicontinuous the graph of T is closed by Proposition 3, and therefore it contains the closure of the restriction of the graph of S_f onto $\text{co } \mathcal{E}(K)$, which coincides with the entire graph of S_f because $\text{co } \mathcal{E}(K)$ is dense in K by the Krein-Milman theorem and S_f is continuous; this proves that S_f is a selection of T . Finally, if two continuous affine maps $a, b : K \rightarrow X$ coincide on $\mathcal{E}(K)$ then they coincide on the convex hull $\text{co } \mathcal{E}(K)$ of $\mathcal{E}(K)$ which is dense in K , so $a = b$ everywhere on K by continuity. Thus every continuous affine selection S of T is uniquely determined by its values on $\mathcal{E}(K)$ (alternatively, we can invoke condition 8 of Theorem 20). The proof of Theorem 21 is complete.

Note that Theorem 21 extends Theorem 17: if K is a Bauer simplex and T is assumed to be an upper semicontinuous convex map acting into arbitrary (not necessarily convex or compact) closed subsets of a Banach space X , this is sufficient for the existence of affine selections of T and even their complete characterization. Taking Proposition 13 into account, this result also extends Theorem 11 in a similar way.

Theorem 22. *Suppose that \mathcal{S} is a semilinear space with a base K which is a metrizable compact set in a locally convex Hausdorff linear topological space, K is also a Bauer simplex, μ_x is the measure on $\mathcal{E}(K)$ with barycenter x for $x \in K$, X is a Banach space, $T : \mathcal{S} \rightarrow 2^X \setminus \{\emptyset\}$ is an upper semicontinuous superlinear set-valued map and values of T are closed in X . Then for every $f \in C(\mathcal{E}(K) \rightarrow X)$ satisfying $f(x) \in T(x)$ for all $x \in \mathcal{E}(K)$ function $S_f(\lambda x) = \lambda \int_{\mathcal{E}(K)} f(y) d\mu_x(y)$, $x \in K$, $\lambda > 0$,*

is a continuous linear selection S of T satisfying $S(x) = f(x)$ for $x \in \mathcal{E}(K)$. Conversely, any continuous linear selection S of T has the form $S = S_f$ where f is the restriction of S onto $\mathcal{E}(K)$.

Theorem 22 follows at once from Proposition 13 and Theorem 21 applied to the restriction of T onto K . These results, however, extend a very special case of the general Theorems 17 and 11, since the semilinear space \mathcal{S} under the conditions of Theorem 22 is isomorphic to $\mathcal{M}_+(\mathcal{E}(K))$ with the weak* topology and the compact set K under the conditions of Theorem 21 is isomorphic to $\mathcal{M}_{+,1}(\mathcal{E}(K))$ with the weak* topology. These observations lead to the following generalization of Theorem 22.

Theorem 23. *Suppose that K is a metrizable compact topological space, X is a Banach space, $T : \mathcal{M}_+(K) \rightarrow 2^X \setminus \{\emptyset\}$ is a superlinear set-valued map, values of T are closed in X and T is upper semi-continuous with respect to the weak* topology of $\mathcal{M}_+(K)$. Then for every $f \in C(K \rightarrow X)$ satisfying $f(x) \in T(\delta_x)$ for all $x \in K$ function $S_f(\mu) = \int_K f(y)d\mu(y)$, $\mu \in \mathcal{M}_+(K)$ is a unique continuous linear selection of T satisfying $S(\delta_x) = f(x)$ for $x \in K$. Conversely, any continuous linear selection S of T has the form $S = S_f$ where $f(x) = S(\delta_x)$ for $x \in K$.*

Indeed, let $\mathcal{K} = \mathcal{M}_{+,1}(K)$ with the weak* topology induced from $\mathcal{M}_+(K)$. Thus \mathcal{K} is a metrizable compact set since K and consequently $C(K)$ are separable. The space $\mathcal{M}_+(K)$ is a lattice: for every $\mu_1, \mu_2 \in \mathcal{M}_+(K)$ both of these measures are absolutely continuous with respect to $\nu = \mu_1 + \mu_2$, thus $d\mu_1 = \sigma_1 d\nu$, $d\mu_2 = \sigma_2 d\nu$ with some $\sigma_1, \sigma_2 \in L_1(\nu)$ and $d(\mu_1 \vee \mu_2) = (\sigma_1 \vee \sigma_2)d\nu$ is the least upper bound for μ_1 and μ_2 . Therefore \mathcal{K} is a simplex. It is easy to see that the map $x \mapsto \delta_x$ establishes a homeomorphism ρ between K and $\mathcal{E}(\mathcal{K})$, thus \mathcal{K} is a Bauer simplex. Moreover, the measure on $\mathcal{E}(\mathcal{K})$ representing any $\mu \in \mathcal{K}$ coincides with $\rho \circ \mu$. Therefore an application of Theorem 22 to the compact set \mathcal{K} yields Theorem 23. Of course, Theorem 23 can also be verified directly without any use or mention of the Choquet theory by repeating the proof of Theorem 21 with necessary changes.

9. SETS CLOSED IN MEASURE

In this section we establish an analogue of Proposition 8 for sets that are convex, bounded and closed in measure in a Banach ideal space of measurable functions, which makes it possible to apply some of the main results of this paper in this somewhat obscure but highly useful setting. For more detail on basic definitions and properties see, e. g., [18].

Let (Ω, μ) be a σ -finite measurable space, and $S(\Omega, \mu)$ be the set of all measurable functions on (Ω, μ) with its usual lattice structure. It is

said that $X \subset S(\Omega, \mu)$ is an *ideal⁸ lattice of measurable functions on the measurable space (Ω, μ)* if X is an ideal in $S(\Omega, \mu)$, i. e. if conditions $f \in X$, $g \in S(\Omega, \mu)$ and $|g| \leq |f|$ a. e. imply that $g \in X$, and the support $\text{supp } X = \bigcup_{f \in X} \text{supp } f$ of X coincides with the entire set Ω up to a set of μ -measure 0. It is said that X is a *normed ideal lattice* of measurable functions on (Ω, μ) if X is an ideal lattice equipped with a norm $\|\cdot\|_X$ such that for any $f \in X$ and $g \in S(\Omega, \mu)$ satisfying $|g| \leq |f|$ one has $\|g\|_X \leq C\|f\|_X$ with some constant C_X independent of f and g . We say that X is a *Banach ideal lattice* of measurable functions on (Ω, μ) if X is a normed ideal lattice that is a Banach space, i. e. if X complete with respect to the norm $\|\cdot\|_X$.

For a Banach ideal lattice X of measurable functions on (Ω, μ) any order continuous functional f on X (order continuity means that given a sequence $x_n \in X$ such that $\sup_n |x_n| \in X$ and $x_n \rightarrow 0$ a. e. it follows that $f(x_n) \rightarrow 0$) has an integral representation $f(x) = \int xy_f$ for some function y_f that is identified with f . The set of all such functionals X' is a Banach ideal lattice with the norm defined by $\|f\|_{X'} = \sup_{g \in X, \|g\|_X=1} \int |fg|$. The lattice X' is called the *order dual* of the lattice X . If X is a Banach ideal lattice then every functional from X' is continuous on X .

A Banach ideal lattice of measurable functions on (Σ, μ) is said to satisfy *the Fatou property* if for any $f_n, f \in X$ such that $\|f_n\|_X \leq 1$ and the sequence f_n converges to f a. e. it is also true that $f \in X$ and $\|f\|_X \leq 1$. We can always assume that $C_X = 1$ if X has the Fatou property. The Fatou property of a lattice X is equivalent to μ -closedness of the unit ball B_X of the lattice X (here and elsewhere μ -convergence denotes convergence in measure in all measurable sets $E \subset \Omega$ satisfying $\mu(E) < \infty$). If X is a Banach ideal lattice then the Fatou property of X is equivalent to order reflexivity of X , i. e. to the relation $X'' = X$. If X is a Banach ideal lattice having the Fatou property then the lattice X' is a norming set of functionals for X , i. e. $\|f\|_X = \sup_{g \in X', \|g\|_{X'}=1} \int fg$ for all $f \in X$.

It is easy to see that μ -closedness in a Banach ideal lattice X satisfying the Fatou property is not weaker than norm-closedness, and it is often strictly stronger. For example, the set of continuous functions f satisfying $|f| \leq 1$ is norm-closed in the lattice $X = L_\infty([0, 1])$ but its closure in measure coincides with the unit ball of X . Sets that are convex, bounded and closed in measure in a Banach ideal lattice X having the Fatou property behave in some ways like compact sets. It is easy to see that the collection of such sets $\mathcal{V}(X)$ contains all compact convex sets of X (in the strong topology of X) since every sequence $f_n \in K$ in a compact set $K \subset X$ converging in measure to some $f \in X$

⁸The word “ideal” in this term is often omitted but understood in the literature working with general lattices of measurable functions; we keep it here to avoid confusion.

has a subsequence converging in X and thus in measure to some $g \in K$ and therefore $f = g \in K$. Also observe that $\mathcal{V}(X)$ is contained in the collection of nonempty convex sets of X that are compact in the weak topology of X since sets from $\mathcal{V}(X)$ are closed in X ; therefore $\mathcal{V}(X)$ is an intermediate space between the spaces of convex sets that are closed in the strong and the weak topologies of X . The following theorem shows that $\mathcal{V}(X)$ has compact type.

Theorem 24 (see [18, Theorem 3, Chapter X, §5]). *Let X be a Banach ideal lattice on (Σ, μ) having the Fatou property and let $\{V_\xi\}_{\xi \in \Xi}$ be a centered family of sets in X that are convex, bounded and μ -closed. Then $\bigcap_{\xi \in \Xi} V_\xi$ is not empty.*

The following proposition that we are going to use repeatedly is, essentially, rather well known (cf., e. g., [20, 1.2], [35, Proposition 15] or [5]).

Proposition 25. *Suppose that X is a Banach lattice of measurable functions on (Ω, μ) having the Fatou property and $F_n \subset X$ is a decreasing sequence of nonempty sets in X that are bounded, convex and μ -closed. Then for any sequence $f_n \in F_n$ there exists a sequence of finite convex combinations*

$$g_n = \sum_{k \geq n} \alpha_k^{(n)} f_k, \quad \alpha_k^{(n)} \geq 0, \quad \sum_{k \geq n} \alpha_k^{(n)} = 1$$

such that g_n converges a. e. to some $f \in F = \bigcap_n F_n$.

In order to prove Proposition 25 we introduce a nonincreasing sequence of sets

$$A_n = \text{clos}_{\mu} \text{co}_{\mu} \{f_k\}_{k \geq n},$$

where clos_{μ} denotes the closure in measure and co_{μ} denotes the convex hull. By the assumptions of the proposition $A_n \subset F_n$ for all n , so Proposition 24 applied to the family $\{A_n\}$ indicates that $\bigcap_n A_n$ is not empty. Take any $f \in \bigcap_n A_n$. Then $f \in F$ and for every n there exists a sequence $g_k^n \in \text{co}_{\mu} \{f_k\}_{k \geq n}$ such that g_k^n converges to f in measure on all sets of finite measure as $k \rightarrow \infty$. Since convergence in measure is metrizable, one can easily find a sequence of indices k_n such that $g_{k_n}^n$ converges to f in measure on all sets of finite measure. Therefore some subsequence of this sequence converges to f almost everywhere, which concludes the proof of Proposition 25.

Proposition 26. *Suppose that X is a Banach ideal lattice of measurable functions on a σ -finite measurable space (Ω, μ) and a semilinear space $\mathcal{V} = \mathcal{V}(X)$ consists of all nonempty bounded convex sets in X that are μ -closed. Then \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(X)$, i. e. \mathcal{V} is closed under the semilinear operations of $\mathcal{C}(X)$. \mathcal{V} has compact type and is order regular. The set X' of all order continuous linear*

functionals on X is exhaustive for X , and all $f \in X'$ are defining for \mathcal{V} . Suppose also that $\Omega = \mathbb{Z}$. Then the set of point evaluation functionals $\mathcal{D} = \{f_j\}_{j \in \mathbb{Z}}$, $f_j(\{x_k\}) = x_j$, is exhaustive for X and consistent with \mathcal{V} .

The proof is quite similar to the proof of Proposition 8. First, suppose that $A, B \in \mathcal{V}$; we need to show that $A + B$ is closed in measure and thus $A + B \in \mathcal{V}$. Let $z \in \text{clos}(A + B)$. Then there exists a sequence $z_n \in A + B$, $z_n = x_n + y_n$, $x_n \in A$, $y_n \in B$, such that $z_n \rightarrow z$ in measure. By Proposition 25 there exists a sequence of finite convex combinations $a_n = \sum_{m \geq n} \alpha_{n,m} x_m$ converging to some $x \in A$ in X . Applying Proposition 25 again to the sequence $b_n = \sum_{m \geq n} \alpha_{n,m} y_m$ allows us to conclude that there exists a sequence of convex combinations $d_n = \sum_{m \geq n} \beta_{n,m} b_m$ converging in X to some $y \in B$. Let $c_n = \sum_{m \geq n} \beta_{n,m} a_m$; then $c_n \in A$, $d_n \in B$, $c_n \rightarrow x$ and $d_n \rightarrow y$ in X , and therefore in measure. At the same time $c_n + d_n$ is a sequence of convex combinations of z_n , so $c_n + d_n \rightarrow z$. This means that $z = x + y$, so $A + B$ is indeed closed in measure and \mathcal{V} is a subspace of $\mathcal{C}(X)$. Compact type of \mathcal{V} follows from Theorem 24. We have $\inf \mathfrak{A} = \bigcap \mathfrak{A}$ for any lower directed family of sets $\mathfrak{A} \subset \mathcal{V}$, and so the rest of order regularity is established exactly as in the proof of Proposition 8. Finally, the set X' is exhaustive for X because X' is norming for X . Let $f \in X'$ be an order continuous functional in X and $A \in \mathcal{V}$. Since A is bounded and f is continuous, $f(A)$ is also bounded and convex. A is closed in X (because it is closed in measure), so by the Open Mapping Theorem $f(A)$ is closed and thus $f(A) = [a, b]$ for some $a, b \in \mathbb{R}$, so f is defining for \mathcal{V} . Let us now verify the claims about the case $\Omega = \mathbb{Z}$. It is obvious that \mathcal{D} is exhaustive for X . Since convergence in measure implies convergence everywhere in Ω , functionals f_j are continuous for all $j \in \mathbb{Z}$ with respect to convergence in measure and thus for every $A \in \mathcal{V}$ and $c \in \mathbb{R}$ nonemptiness of $f^{-1}(c) \cap A$ implies that $f^{-1}(c) \cap A$ is closed in measure and nonempty, so $f^{-1}(c) \cap A \in \mathcal{V}$ and f_j is consistent with \mathcal{V} . The proof of Proposition 26 is complete.

Proposition 26 makes it possible to apply the main results of this paper to certain situations lacking compactness, for example, to bounded sets in $X = l^1$ (and other non-dual lattices) that are closed in measure. Note that in general lattices order continuous functionals may not be consistent with \mathcal{V} defined in Proposition 26; for example, the mean $f(x) = \int_{\Omega} x(\omega) d\omega$ is an order continuous functional in $X = L_1([0, 1])$, but it is easy to see that

$$0 \in (\text{clos } f^{-1}(1) \cap B_{L_1([0,1])}) \setminus (f^{-1}(1) \cap B_{L_1([0,1])}),$$

where $B_{L_1([0,1])}$ denotes the closed unit ball of X . In fact, it is easy to see that no nontrivial order continuous functional on $X = L_1([0, 1])$ is consistent with $\mathcal{V} = \mathcal{V}(X)$ described in Proposition 26. However, another technique of obtaining linear selections allows us to extend Theorem 11 to the case of maps acting on the cone of nonnegative functions in L_1

into $\mathcal{V}(X)$ under additional assumptions; see Theorem 38 in Section 12. It is unclear whether the conclusion of Theorem 11 holds true for this particular space $\mathcal{V}(X)$ under general conditions or for general Banach ideal lattices.

10. SEMILINEAR SPACES WITH CONE-BASES

In [40] and [41] some results on additive and affine selections were obtained for maps defined on a cone with a cone-basis. In this section we will show how the main results of this paper become very clear and instantaneous under the assumption that the semilinear space the maps are acting on has a cone-basis. However, this assumption is stronger than the interpolation property, and in Section 11 below it will become clear that it is often violated outside the case of cones in a finite-dimensional space.

A set $B \subset K$ in a semilinear space K is said to be a *cone-basis* of K if for every $x \in K$ there exist some unique nonnegative coordinates $\alpha_b(x) \geq 0$ that are zero for all $b \in B$ except for a finite number and $x = \sum_{b \in B} \alpha_b(x)b$.

Proposition 27. *Any cone with a cone-basis has the interpolation property.*

Indeed, suppose that $z \in K$ and we are given some

$$\{x_j\}_{j=1}^m, \{y_k\}_{k=1}^n \subset K$$

satisfying $z = \sum_{j=1}^m x_j = \sum_{k=1}^n y_k$. Then

$$z = \sum_{b \in B} \alpha_b(z)b, \quad x_j = \sum_{b \in B} \alpha_b(x_j)b, \quad y_k = \sum_{b \in B} \alpha_b(y_k)b.$$

Let $z_{jk} = \sum_{b \in B} \frac{\alpha_b(x_j)\alpha_b(y_k)}{\alpha_b(z)}b$; here we use the usual convention $\frac{0}{0} = 0$ (note that $\alpha_b(z) = 0$ implies $\alpha_b(x_j) = 0$ and $\alpha_b(y_k) = 0$). Surely, $x_j = \sum_{k=1}^n z_{jk}$ and $y_k = \sum_{j=1}^m z_{jk}$ for all j and k , so the interpolation property of K is established.

On the other hand, not every cone with the interpolation property has a cone-basis. In fact, Theorem 32 in Section 11 below indicates that in no infinite-dimensional normed ideal lattice of measurable functions the cone of nonnegative functions has a cone-basis. The following proposition gives a very easy particular case of Theorem 32.

Proposition 28. *The cone $K = [L_1([0, 1])]_+$ of nonnegative summable functions on the segment $[0, 1]$ has the interpolation property, but it has no cone-basis.*

The cone K has the interpolation property as the cone of positive functions in a lattice. Let us show that K does not have a cone-basis. Indeed, suppose that $B \subset K$ is a cone-basis of K . We can assume by renorming that $\|b\|_{L_1} = 1$ for all $b \in B$. Then every function $z \in B$ has

to be an extreme point of the unit ball of L_1 : if $z = (1 - \theta)x + \theta y$ for some $0 < \theta < 1$ and $x, y \in L_1$ such that $\|x\|_{L_1} \leq 1$ and $\|y\|_{L_1} \leq 1$, then we can make x and y positive (and thus belonging to K) by replacing them with $x \vee 0$ and $y \vee 0$ respectively, and the definition of a cone-basis implies that $\alpha_b(z) = (1 - \theta)\alpha_b(x) + \theta\alpha_b(y)$ for all $b \in B$, which means that $x = y = b$. But it is well known that the unit ball of L_1 has no extreme points, so K does not have a cone-basis.

This example shows that the existence of a cone-basis for cones in infinite-dimensional spaces is not that common. In fact, in Theorem 32 below we show that the only ideal normed lattices in which the cone of nonnegative functions has a cone-basis are finite-dimensional lattices. This contrasts starkly with the fact that every linear space has a Hamel (i. e. algebraic) basis. However, the existence of a cone-basis allows one to parametrize the set of all linear selections of a superlinear set-valued map with almost no effort. The following proposition follows immediately from the definitions.

Proposition 29. *Suppose that \mathcal{S} is a semilinear space having a cone-basis B and $T : \mathcal{S} \rightarrow \mathcal{V}$ is a superlinear map acting into a semilinear space \mathcal{V} equipped with a transitive relation \leq compatible with the semilinear structure of \mathcal{V} . Then for every $f : B \rightarrow \mathcal{V}$ satisfying $f \leq T$ on the set B the map defined by $S_f(x) = \sum_b \alpha_b(x)f(b)$ is a linear submap $S_f \leq T$ of T such that $S_f \leq f$ on B . In particular, if f is the restriction of T on B then $S_T = S_f$ is the greatest linear submap of T .*

The next proposition strengthens Theorem 9 in the case when the space \mathcal{S} has a cone-basis.

Proposition 30. *Suppose that \mathcal{S} and \mathcal{W} are semilinear spaces, \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(\mathcal{W})$ and \mathcal{S} has a cone-basis B . Let $A : \mathcal{S} \rightarrow \mathcal{V}$ be a linear map. Then for any $x \in \mathcal{S}$ and $y \in A(x)$ there exists a linear selection $a : \mathcal{S} \rightarrow \mathcal{W}$ of A satisfying $a(x) = y$.*

Indeed, let $x = \sum_{b \in B} \alpha_b(x)b$, and $B_x = \{b \in B \mid \alpha_b(x) \neq 0\}$. Then $A(x) = \sum_{b \in B_x} \alpha_b(x)A(b)$ by linearity of A , so $y = \sum_{b \in B_x} \alpha_b(x)y_b$ with some $y_b \in A(b)$ for $b \in B_x$. Select arbitrary $y_b \in A(b)$ for all $b \in B \setminus B_x$ (possibly making use of the Axiom of Choice) and define the mapping a by $a(z) = \sum_{b \in B} \alpha_b(z)y_b$ for any $z = \sum_{b \in B} \alpha_b(z)b$. Then

$$a(z) \in \sum_{b \in B} \alpha_b(z)A(b) = A\left(\sum_{b \in B} \alpha_b(z)b\right) = A(z)$$

and $a(x) = y$, which concludes the proof of Proposition 30.

Finally, it is easy to obtain from the previous two propositions a version of Theorem 11 for the case when space \mathcal{S} has a cone-basis.

Proposition 31. *Suppose that \mathcal{S} and \mathcal{W} are semilinear spaces, \mathcal{S} has a cone-basis B and \mathcal{V} is a subspace of the semilinear space $\mathcal{C}(\mathcal{W})$. Let*

$T : \mathcal{S} \rightarrow \mathcal{V}$ be a superlinear map. Then the following conditions are equivalent.

- (1) There exists a linear selection a of T satisfying $a(x) = y$.
- (2) $y \in \sum_{b \in B} \alpha_b(x) T(b)$.

Note that in the finite-dimensional case by the Krein-Milman theorem a compact convex set $K \subset \mathbb{R}^n$ is a Choquet (or Baire) simplex if and only if the set $\mathcal{E}(K)$ of the extreme points of K is a cone-basis in K_{sus} . Thus it is possible to obtain Theorems 18 and 20 in the case $K \subset \mathbb{R}^n$ (or, equivalently, $\mathcal{S} \subset \mathbb{R}^{n+1}$) from the simple results described in this section.

11. SHARPNESS OF THE COMPACT TYPE CONDITION

In this section we present an example that shows that the requirement for \mathcal{V} to have compact type cannot be dropped from the conditions of Theorem 9. We also prove, using essentially the same idea, that the cone of positive functions in an ideal normed lattice of measurable functions on a measurable space (Ω, μ) has no cone-basis unless measure μ is discrete.

Let $\mathcal{S} = [l^\infty]_+$ be the set of bounded sequences of nonnegative real numbers, c_0 be the set of sequences that converge to 0 with the usual norm $\|\{a_n\}\|_{c_0} = \sup_n |a_n|$, $\mathcal{W} = [c_0]_+$ be the cone of sequences of nonnegative numbers in c_0 and $\mathcal{V} \subset \mathcal{C}(\mathcal{W})$ be the set of all nonempty bounded convex closed sets in \mathcal{W} . It is easy to see that \mathcal{V} does not have compact type: take, for example, a diverging sequence a bounded by 1 and a decreasing sequence of sets

$$A_n = \{\{b_j\} \in \mathcal{W} \mid |b_j| \leq 1 \text{ for all } j, b_j = a_j \text{ for } 1 \leq j \leq n\}$$

that all lie in \mathcal{V} but their intersection $\bigcap_n A_n = \{a\}$ is not a subset of \mathcal{W} . However, \mathcal{V} is a subspace of $\mathcal{C}(\mathcal{W})$ that has an exhaustive set of linear functionals $f_j(\{a_k\}) = a_j$ which are defining for and consistent with \mathcal{V} . Let us define a map $A : \mathcal{S} \rightarrow \mathcal{V}$ by $A(\{a_j\}) = \mathcal{W} \cap \prod_j [0, a_j]$. Linearity of T is easily verified, but, as we will now show, for any sequence $x = \{x_j\} \in c_0$ such that $x_j > 0$ for all $j \in \mathbb{N}$ there is no linear selection a of A satisfying $a(x) = x$. Suppose, on the contrary, that there is such a selection a . For any $I \subset \mathbb{N}$ we denote by χ_I the indicator function for I on \mathbb{N} . As $a(\chi_I x) \leq \chi_I x$ for any $I \subset \mathbb{N}$, it follows from additivity of a that $a(\chi_I x) = \chi_I x$ for any $I \subset \mathbb{N}$. Let $e_j = \chi_{\{k \mid k=j\}} \in c_0$ be the usual coordinate vectors. Then $a(e_j) = e_j$, so for any finite $I \subset \mathbb{N}$ and any $z \in \mathcal{S}$ we have $a(\chi_I z) = \chi_I z$, and $a(z) = a(\chi_I z) + a(\chi_{\mathbb{N} \setminus I} z) \geq \chi_I z$. Therefore $a(z) = z$ for all $z \in \mathcal{S}$, a contradiction since, for example, $z = \chi_{\mathbb{N}} \notin \mathcal{W}$. Thus we have constructed an example of semilinear spaces \mathcal{S} , \mathcal{W} , \mathcal{V} and a map A that satisfy all conditions of Theorem 9 except for the compact type of \mathcal{V} and for which the conclusion of Theorem 9 does not hold.

Let ν be the counting measure on \mathbb{N} . The construction just described actually shows that for no Banach ideal lattice X on the measurable space (\mathbb{N}, ν) the cone $\mathcal{S} = X_+$ of nonnegative functions has a cone-basis. Let \mathcal{W} be any ideal lattice of measurable functions which is a proper sublattice of X . We can take, for example,

$$\mathcal{W} = \{\{a_n\} \in X \mid \{na_n\} \in X\},$$

which is a Banach ideal lattice with respect to the norm

$$\|\{b_n\}\|_{\mathcal{W}} = \|\{na_n\}\|_X.$$

It is easy to see that \mathcal{W} is a proper sublattice of X : for example, sequence $n^{-\frac{1}{2}} \frac{e_n}{\|e_n\|_X}$ converges to 0 in X but diverges in \mathcal{W} . Suppose that $\mathcal{S} = X_+$ has a cone-basis. Then by Proposition 30 map A would have had a linear selection a satisfying $a(x) = x$ for some $x \in \mathcal{W}_+$. There exists some $x \in \mathcal{W}_+$ such that $\text{supp } x = \mathbb{N}$; for example, $x = \left\{ \frac{2^{-n}}{\|e_n\|_{\mathcal{W}}} \right\}$. Then the argument from the example shows that $a(x) = x$ for all $x \in \mathcal{S}$ and thus $X_+ = \mathcal{W}_+$, a contradiction. Therefore for no Banach ideal lattice X on (\mathbb{N}, ν) the cone X_+ has a cone-basis. This result can be extended to all nondiscrete measurable spaces.

Theorem 32. *Suppose that X is a normed ideal lattice of measurable functions on a σ -finite measurable space (Ω, μ) . The following conditions are equivalent.*

- (1) *The cone of nonnegative functions X_+ in X has a cone-basis.*
- (2) *Measure μ is discrete, i. e. μ is a finite sum of point masses*

$$\mu = \sum_{j=1}^N c_j \delta_{\omega_j}, \quad \omega_j \in \Omega.$$
- (3) *X is finite-dimensional.*

Implication $2 \Rightarrow 1$ is straightforward: if condition 2 is satisfied then $W = \{\chi_{\omega_j}\}_{j=1}^N$ is a cone-basis for X_+ . Equivalence of conditions 2 and 3 is also simple. If condition 2 is satisfied then W is a basis for X , so implication $2 \Rightarrow 3$ holds true. On the other hand, if measure μ is nondiscrete then there exists a countable family $\{E_j\}_{j \in \mathbb{N}}$ of disjoint measurable sets in Ω having positive finite measure, and with it an infinite family of functions from X supported in E_j that is linearly independent, so if condition 2 is not satisfied then condition 3 is also not satisfied, which proves implication $3 \Rightarrow 2$. Let us establish the remaining implication $1 \Rightarrow 2$. Suppose that it does not hold for some measure μ , i. e. μ is nondiscrete but X_+ has a cone-basis; we will now devise a construction from these two assumptions that would lead to a contradiction. Since measure μ is nondiscrete and σ -finite there exists a countable partition $\mathcal{E} = \{E_n\}_{n \in \mathbb{N}}$ of Ω (i. e. sets E_n are pairwise disjoint and $\bigcup_n E_n = \Omega$) such that $\chi_{E_n} \in X$ (see, e. g., [19, Chapter IV, §3, Corollary 2] or [18]). If w is a measurable function which is positive a. e. the weighted normed lattice $X(w)$ is defined by $X(w) = \{fw \mid f \in X\}$ with the weighted norm $\|g\|_{X(w)} = \|gw^{-1}\|_X$. Let $w = \sum_{n \in \mathbb{N}} 2^{-n} \chi_{E_n}$,

$\mathcal{W} = [X(w)]_+$ and $\mathcal{S} = X_+$. It is easy to see that $X(w) \subset X$ (continuous inclusion) because $w \leq 1$, and therefore $\mathcal{W} \subset \mathcal{S}$. Define $\mathcal{V} \subset \mathcal{C}(\mathcal{W})$ by $\mathcal{V} = \{\{f \in \mathcal{W} \mid 0 \leq f \leq a\} \mid a \in \mathcal{W}\}$, and define a map $A : \mathcal{S} \rightarrow \mathcal{V}$ by $A(g) = \{f \in \mathcal{W} \mid 0 \leq f \leq g\}$. Let $g = \sum_{n \in \mathbb{N}} 2^{-n} \frac{\chi_{E_n}}{\|\chi_{E_n}\|_{X(w)}}$. Since by assumptions $X_+ = \mathcal{S}$ has a cone-basis, by Proposition 30 there exists a linear selection a of A satisfying $a(g) = g$. Observe that for any $f \in X_+$ we have $a(f) \leq f$. Therefore, for any measurable $E \subset \Omega$ we have $a(g\chi_E) \leq g\chi_E$ and $a(g\chi_{\Omega \setminus E}) \leq g\chi_{\Omega \setminus E}$. But $g = g\chi_E + g\chi_{\Omega \setminus E}$ and $a(g) = g$, so $a(g\chi_E) = g\chi_E$; in particular, $a(\chi_{E_n}) = \chi_{E_n}$ for any n . Let $h = \sum_{n \in \mathbb{N}} 2^{-\frac{n}{2}} \frac{\chi_{E_n}}{\|\chi_{E_n}\|_X}$. It is easy to see that $h \in \mathcal{S}$. By the assumptions, $a(h) \in \mathcal{W}$, but

$$\begin{aligned}
 \|a(h)\|_{X(w)} &= \|a(h\chi_{E_n}) + a(h\chi_{\Omega \setminus E_n})\|_{X(w)} \geq \|a(h\chi_{E_n})\|_{X(w)} = \\
 &= \left\| \frac{2^{-\frac{n}{2}}}{\|\chi_{E_n}\|_X} a(\chi_{E_n}) \right\|_{X(w)} = \left\| \frac{2^{-\frac{n}{2}}}{\|\chi_{E_n}\|_X} \chi_{E_n} \right\|_{X(w)} = \\
 &= \left\| \frac{2^{\frac{n}{2}}}{\|\chi_{E_n}\|_X} \chi_{E_n} \right\|_X = 2^{\frac{n}{2}}
 \end{aligned}$$

for any $n \in \mathbb{N}$, so $a(h) \notin \mathcal{W}$. This contradiction concludes the proof of Theorem 32.

12. LINEAR AND SUPERLINEAR SET-VALUED MAPS OF $[L_1]_+$

In Section 8 above we have seen that for upper semicontinuous superlinear maps acting on the semilinear space of Borel measures $\mathcal{M}_+(K)$ on a compact set K the conditions necessary for the existence of linear selections are significantly weaker compared to the general case treated in the main results of this paper, and indeed we have a natural parametrization of the set of all linear selections in this case. The semilinear space that often arises in applications, however, is the cone $\mathcal{S} = [L_1]_+$ of measurable functions on some measurable space (Ω, μ) with a σ -finite measure μ that are positive a. e. and summable. In this section we will see how in certain cases the main results of this survey can be extended in this setting; the most useful, perhaps, is a simple construction proving the existence of a linear selection for every bounded linear map $T : [L_1]_+ \rightarrow \mathcal{C}(X)$ acting into a Banach space X if the measurable space Ω is separable. Additional assumptions provide a result similar to Theorem 9 for the case of maps taking values in sets closed in measure.

Let \mathcal{S} be a monoid semilinear space. We say that

$$\mathcal{B} = \{b_{n,k}\}_{\substack{n \geq 0, \\ 0 \leq k < 2^n}} \subset \mathcal{S}$$

is a *nesting basis* of \mathcal{S} if it satisfies the following conditions.

- (1) $b_{n,k} = b_{n+1,2k} + b_{n+1,2k+1}$ for all n and k .

- (2) $b_{n_1, k_1} = b_{n_2, k_2}$ implies that either $n_1 = n_2$ and $k_1 = k_2$ or $b_{n_1, k_1} = b_{n_2, k_2} = 0$.
- (3) The conic hull of \mathcal{B} coincides with \mathcal{S} .

Let us give an example right away. Suppose that $\mathfrak{A} = (\Omega, \Sigma, \mu)$ is a separable measurable space. This means that there exists a sequence of sets $\{A_j\}_{j \geq 1} \subset \Sigma$ such that the minimal σ -algebra $\Sigma_0 = \sigma(\{A_j\}_{j \geq 1})$ containing all of them is μ -dense in \mathfrak{A} . By renorming measure μ if necessary we can assume that $\mu(\Omega) = 1$. We will now construct a nesting basis for the space \mathcal{S} of simple functions in $[L_1(\Omega, \Sigma_0, \mu)]_+$. Let $b_{0,0} = \chi_\Omega$ and $\Sigma_n = \sigma(\{A_j\}_{1 \leq j < 2^n})$; this σ -algebra consists from disjoint unions of at most 2^n atoms that we are going to denote by $B_{n,k}$, $0 \leq k < 2^n$, possibly with empty sets in certain places. We can also assume that $B_{n+1,2k} \cup B_{n+1,2k+1} \subset B_{n,k}$ for all n and k . Then functions $b_{n,k} = \chi_{B_{n,k}}$ form a nesting basis. For example, for the unit segment $\Omega = [0, 1]$ with a Lebesgue measure we can take the usual dyadic partition $B_{n,k} = [2^{-n}k, 2^{-n}(k+1)]$.

Proposition 33. *Suppose that \mathcal{S} is a monoid semilinear space that has a nesting basis $\mathcal{B} = \{b_{n,k}\}_{\substack{n \geq 0, \\ 0 \leq k < 2^n}}$, \mathcal{W} is a semilinear space, let $T : \mathcal{S} \rightarrow 2^{\mathcal{W}} \setminus \{\emptyset\}$ be a linear map and $y_0 \in T(b_{0,0})$. Then T has a linear selection satisfying $S(b_{0,0}) = y_0$.*

Indeed, we can define a partial additive selection $S_0 : \mathcal{B} \rightarrow \mathcal{W}$ of T inductively by selecting $S_0(b_{0,0}) = y_0 \in T(b_{0,0})$ and

$$(S_0(b_{n+1,2k}), S_0(b_{n+1,2k+1})) \in \{(g, h) \mid g \in T(b_{n+1,2k}), h \in T(b_{n+1,2k+1}), g + h = S_0(b_{n,k})\};$$

this set is nonempty because $T(b_{n+1,2k}) + T(b_{n+1,2k+1}) = T(b_{n,k})$ due to the linearity of T . We then define a suitable linear selection S of T by

$$S\left(\sum_{j=1}^N \alpha_j b_{n_j, k_j}\right) = \sum_{j=1}^N \alpha_j S_0(b_{n_j, k_j})$$

for all finite sequences of n_j, k_j and $\alpha_j \geq 0$.

Let X and Y be normed spaces, and $\mathcal{S} \subset X$. We say that a map $T : \mathcal{S} \rightarrow 2^Y$ is *bounded* if $\sup_{y \in T(x)} \|y\|_Y \leq C\|x\|_X$ for all $x \in \mathcal{S}$ with some constant C independent of x . Boundedness sometimes allows us to extend selections from dense sets.

Theorem 34. *Suppose that X is a Banach space, (Ω, μ) is a separable measurable space and $T : [L_1(\Omega, \mu)]_+ \rightarrow 2^X$ is a bounded upper semicontinuous linear map. Then T has a continuous linear selection.*

Indeed, we use the construction before Proposition 33 to obtain a σ -algebra (Ω, Σ_0, μ) that is μ -dense in (Ω, Σ, μ) and a nesting basis \mathcal{B} for the semilinear space \mathcal{S}_0 of simple functions on $[L_1(\Omega, \Sigma_0, \mu)]_+$. By Proposition 33 the restriction of T onto \mathcal{S}_0 has a linear selection

$S_0 : \mathcal{S}_0 \rightarrow X$ which is bounded. By Proposition 1 the map S_0 is naturally extended to a bounded linear selection S_1 of T on the lattice $L_1(\Omega, \Sigma_0, \mu)$. Then S_1 can be extended by continuity to a linear bounded and hence continuous map $S : L_1(\Omega, \mu) \rightarrow X$ which we restrict onto the cone $\mathcal{S} = [L_1(\Omega, \mu)]_+$; the graph of S is the closure of the graph of S_1 in $\mathcal{S} \times X$. By Proposition 3 the graph of T is closed, so the graph of S is contained within the graph of T , which means that S is a bounded linear selection of T .

In order to obtain an analogue of Theorem 9 we will need the following proposition, which shows that at least in the present setting upper semicontinuous linear maps also possess a sort of lower semicontinuity.

Proposition 35. *Suppose that (Ω, μ) is a measurable space, Y is a Banach space and $T : [L_1(\Omega, \mu)]_+ \rightarrow 2^Y$ is a bounded linear map. Then for every $f \in [L_1(\Omega, \mu)]_+$, $g \in T(f)$ and a sequence $f_n \in [L_1(\Omega, \mu)]_+$ converging to f in $L_1(\Omega, \mu)$ there exists a sequence $[L_1(\Omega, \mu)]_+$ of convex combinations of $\{f_j\}_{j \geq n}$ and $g_n \in T(h_n)$ such that $h_n \rightarrow f$ in $L_1(\Omega, \mu)$ and $g_n \rightarrow g$ in Y .*

Indeed, let $\varepsilon_n \rightarrow 0$. It suffices to find a convex combination h_n of $\{f_j\}_{j \geq n}$ and $g_n \in T(h_n)$ satisfying $\|g_n - g\|_Y \leq \varepsilon_n$. Suppose that no such h_n and g_n exist; this means that the distance between the closed convex hull G_n of $\{T(f_j) \mid j \geq n\}$ and g is at least ε_n , so $g \notin G_n$. By a separation theorem there exists some bounded functional $\varphi \in Y^*$ on Y satisfying

$$(7) \quad \sup\{\varphi(g) \mid g \in G_n\} < \varphi(g).$$

By Proposition 7 support function φ_T^* defined by $\varphi_T^*(x) = \sup_{y \in T(x)} \varphi(y)$ is bounded and linear. Proposition 1 allows us to extend φ_T^* boundedly onto $L_1(\Omega, \mu)$, and therefore φ_T^* is continuous. This contradicts (7) since the right-hand part is bounded from above by $\varphi_T^*(f)$ but at the same time the left-hand part is at least $\liminf_n \varphi_T^*(f_n) = \varphi_T^*(f)$. The proof of Proposition 35 is complete.

We also need the following refinement of Proposition 33.

Proposition 36. *Suppose that \mathcal{S} is a monoid semilinear space that has a nesting basis $\mathcal{B} = \{b_{n,k}\}_{\substack{n \geq 0, \\ 0 \leq k < 2^n}}$, \mathcal{W} is a semilinear space and $T : \mathcal{S} \rightarrow 2^{\mathcal{W}} \setminus \{\emptyset\}$ is a linear map. Then for every $x \in \mathcal{S}$ and $y \in T(x)$ there exists a linear selection S of T satisfying $S(x) = y$.*

Indeed, we can write

$$(8) \quad x = \sum_{j=1}^N \alpha_j b_{n_j, k_j}$$

with $\alpha_j > 0$ and an additional assumption that

$$(9) \quad b_{n_j, k_j} \prec b_{n_l, k_l} \text{ for no } j \text{ and } l;$$

here \prec is the intrinsic order of the semilinear space \mathcal{S} introduced in Section 1. Otherwise b_{n_l, k_l} is expressed as a sum of b_{n_j, k_j} and a finite number of elements from \mathcal{B} that we can substitute into (8); it is easy to see that every such substitution decreases the total number of relations (9) and so after a finite number of such substitutions condition (9) is satisfied. Similarly, we also express $x_r = b_{0,0} - \sum_{j=1}^N b_{n_j, k_j}$ as $x_r = \sum_{j=N+1}^M b_{n_j, k_j}$. Because of the linearity of T there exist some $y_{n_j, k_j} \in T(b_{n_j, k_j})$ satisfying $y = \sum_{j=1}^N \alpha_j y_{n_j, k_j}$. We then apply Proposition 33 to subspaces $\mathcal{S}_{n_j, k_j} = \{\lambda z \in \mathcal{S} \mid z \preceq b_{n_j, k_j}, \lambda > 0\}$ with their respective bases

$$\mathcal{B}_{n_j, k_j} = \{b \in \mathcal{B} \mid b \preceq b_{n_j, k_j}\}$$

to obtain for all $1 \leq j \leq M$ linear selections S_{n_j, k_j} of T restricted onto \mathcal{S}_{n_j, k_j} satisfying $S_{n_j, k_j}(b_{n_j, k_j}) = y_{n_j, k_j}$ for $1 \leq j \leq N$. It is easy to see that the conic hull of $\bigcup_{j=1}^M \mathcal{S}_{n_j, k_j}$ is the entire \mathcal{S} . Now we can define a linear selection S of T by $S(z) = \sum_{j=1}^M S_{n_j, k_j}(z)$, $z \in \mathcal{S}$, which by the construction satisfies $S(x) = y$.

Theorem 37. *Suppose that X is a Banach ideal lattice of measurable functions on a σ -finite measurable space (Ω_Y, ν) , X satisfies the Fatou property, (Ω, μ) is a separable measurable space, the semilinear space \mathcal{V} consists of all nonempty bounded convex sets in X that are ν -closed and $T : [L_1(\Omega, \mu)]_+ \rightarrow \mathcal{V}$ is a bounded upper semicontinuous linear map. Then for any $f \in [L_1(\Omega, \mu)]_+$ and $g \in T(f)$ there exists a continuous linear selection S of T satisfying $S(f) = g$.*

Indeed, suppose that under the conditions of Theorem 37 we are given some $f \in [L_1(\Omega, \mu)]_+$ and $g \in T(f)$. Following the proof of Theorem 34, there exists a σ -algebra (Ω, Σ_0, μ) that is μ -dense in (Ω, Σ, μ) and a nesting basis \mathcal{B} for the semilinear space \mathcal{S}_0 of simple functions on $[L_1(\Omega, \Sigma_0, \mu)]_+$. Then there exists a sequence $f_n \in \mathcal{S}_0$ converging to f in L_1 . By Proposition 35 there exists a sequence $h_n \in [L_1(\Omega, \mu)]_+$ of convex combinations of $\{f_j\}_{j \geq n}$ and $g_n \in T(h_n)$ such that $h_n \rightarrow f$ in $L_1(\Omega, \mu)$ and $g_n \rightarrow g$ in X . Then by Proposition 36 there exist linear selections S_n of T restricted onto \mathcal{S}_n satisfying $S_n h_n = g_n$, which by Proposition 1 can be extended by continuity to bounded maps on the entire $L_1(\Omega, \mu)$, which are uniformly bounded because they are selections of a bounded map T . Then it is evident that the family $\{S_n\}$ is uniformly continuous. We define a map $R : [L_1(\Omega, \mu)]_+ \rightarrow \mathcal{V}$ by $R_n(x) = \text{clos co}\{S_j(x) \mid j \geq n\}$, $R(x) = \bigcap_n R_n(x)$ for $x \in [L_1(\Omega, \mu)]_+$, where clos denotes ν -closure. Then R is a submap of T having nonempty values belonging to \mathcal{V} because of the compact type of \mathcal{V} guaranteed by Proposition 26. We may assume that $\nu(\Omega_Y) < \infty$ by replacing the measure ν by an equivalent

one. It is well-known that convergence in measure is metrizable; let

$$(10) \quad \delta(x_1, x_2) = \int_{\Omega_Y} \frac{|x_1 - x_2|}{1 + |x_1 - x_2|} d\nu, \quad x_1, x_2 \in X$$

be a translation-invariant metric for this convergence in X . It is also well-known that convergence in X implies convergence in measure for normed ideal lattices X ; see, e. g., [19, Chapter IV, §3, Theorem 1] or [18]. Therefore the inclusion $X \rightarrow X_\delta$ is continuous and thus bounded, where X_δ denotes the space X with metric δ ; together with translation invariance of δ this means that there exists a constant C_X such that $\delta(x_1, x_2) \leq C_X \|x_1 - x_2\|_X$ for all $x_1, x_2 \in X$. We are now going to verify that $R(f) = \{g\}$. Take arbitrary $\varepsilon > 0$. There exists some $N \in \mathbb{N}$ such that $\|f_n - f\|_{L_1} \leq \varepsilon$ and $\|g_n - g\|_X \leq \varepsilon$ for all $n \geq N$. Then $\|S_n f - g_n\|_X = \|S_n f - S_n f_n\|_X \leq C\varepsilon$ for all $n \geq N$ with some constant C independent of ε by the uniform boundedness of $\{S_n\}$, and therefore $\|S_n f - g\|_X \leq \|S_n f - g_n\|_X + \|g_n - g\|_X \leq (C + 1)\varepsilon$ for all $n \geq N$. This implies that $\delta(S_n f, g) \leq C_X(C + 1)\varepsilon$ for $n \geq N$ and therefore by the convexity of (10) we also have $\sup_{y \in R_n(f)} \delta(y, g) \leq C_X(C + 1)\varepsilon$ for all $n \geq N$. Since ε is arbitrary, this means that $R(f) = \{g\}$ as claimed. It is easy to see that the graphs of R_n are closed in $[L_1(\Omega, \mu)]_+ \times X_\delta$, thus the graphs of R_n are closed in a stronger topology $[L_1(\Omega, \mu)]_+ \times X$. Therefore the graph of R is closed in $[L_1(\Omega, \mu)]_+ \times Y$, and by Proposition 3 map R is upper semicontinuous. Finally, we apply Theorem 34 to obtain a continuous linear selection S of R . Then $S(f) = g$ and S is a selection of T . The proof of Theorem 37 is complete.

Finally, we present a version of Theorem 11 for the case treated in this section.

Theorem 38. *Suppose that X is a Banach ideal lattice of measurable functions on a σ -finite measurable space (Ω_Y, ν) , X satisfies the Fatou property, (Ω, μ) is a separable measurable space, the semilinear space \mathcal{V} consists of all nonempty bounded convex sets in X that are ν -closed and $T : [L_1(\Omega, \mu)]_+ \rightarrow \mathcal{V}$ is a bounded upper semicontinuous superlinear map. Then T admits a continuous linear selection. Suppose, additionally, that $x \in [L_1(\Omega, \mu)]_+$ and $y \in T(x)$. Then the following conditions are equivalent.*

- (1) *There exists a continuous linear selection a of T such that $a(x) = y$.*
- (2) *$y \in \sum_{j=1}^N T(x_j)$ for any $N \in \mathbb{N}$ and $\{x_j\}_{j=1}^N \subset [L_1(\Omega, \mu)]_+$ satisfying $x = \sum_{j=1}^N x_j$.*

Indeed, by Theorem 2 (applicable due to Proposition 26 and the space $[L_1(\Omega, \mu)]_+$ being a lattice and thus having the Riesz Decomposition Property) map T has a greatest linear selection

$$S : [L_1(\Omega, \mu)]_+ \rightarrow \mathcal{V}$$

defined by the formula (1) which is upper semicontinuous by Proposition 4. The claims of Theorem 38 then follow from Theorem 37 applied to the map S .

We mention without going into details that the case of linear maps acting on $[L_1]_+$ treated in this section is closely related to the theory of set-valued measures and set-valued integrals that has been fairly extensively developed in the literature; see, e. g., [6] and references contained therein. A linear set-valued map T on $[L_1]_+$ acting into a linear space X naturally corresponds to a set-valued measure $M(A) = T(\chi_A)$, and at least under some additional assumptions linear selections of T can be recovered from selections of M that can in some cases be described as measurable selections f of a Radon-Nikodým derivative F of M , which provides a parametrization of the set of suitable linear selections S of T of the form $S_f(g) = \int g df$ in the spirit of Section 8 and also provides additional sufficient conditions for the existence of linear selections of T . However, this approach requires that X has the Radon-Nikodým property, which is notoriously absent in the case $X = L_1$ that usually appears in the applications, so we will not pursue it in the present work.

13. ADDITIVITY AND LINEARITY

In this short section we give some remarks about rather close relationship between additivity and linearity for maps between semilinear spaces. Similar results are discussed in [40] (see also [41]). This might be useful in applications if a set-valued map under study is merely superadditive but we want to know if its additive selections are superlinear.

It is easy to see that if $A : \mathcal{S} \rightarrow \mathcal{V}$ is an additive map acting between two semilinear spaces \mathcal{S} and \mathcal{V} then $A(kx) = kA(x)$ for any positive integer k , hence $A(\lambda x) = \lambda A(x)$ for all $x \in \mathcal{S}$ and positive rational $\lambda \in \mathbb{Q}_+$. This property is called \mathbb{Q}_+ -homogeneity. Since \mathbb{Q}_+ is dense in \mathbb{R}_+ this property is often as good as the usual homogeneity. However, it is well known that there exists an additive function $T : \mathbb{R} \rightarrow \mathbb{R}$ which is not linear (see, e. g., [14, §2.26], disregard the slight confusion of terms). Equipping $\mathcal{V} = \mathbb{R}$ with the trivial partial order ($x \leq y$ if and only if $x = y$) shows that in Theorem 2 the map $S = T$ might not be linear despite being \mathbb{Q}_+ -linear. However, introduction of certain rather weak continuity assumptions remedies the situation.

Proposition 39. *Suppose that \mathcal{S} and \mathcal{V} are semilinear spaces and $A : \mathcal{S} \rightarrow \mathcal{V}$ is an additive map. Suppose also that \mathcal{V} is a monoid semilinear space equipped with a Hausdorff topology such that the semilinear operations of \mathcal{V} are continuous, and for every $x \in \mathcal{S}$ the map $\nu \mapsto A(\nu x)$ is bounded at some $\alpha \geq 0$, i. e. for every neighbourhood U*

of 0 in \mathcal{V} there exist some $m > 0$ and $\varepsilon > 0$ such that $A(\nu x) \in mU$ for $|\nu - \alpha| < \varepsilon$. Then A is linear.

Suppose that $\lambda > 0$ and $x \in \mathcal{S}$; we need to prove that $A(\lambda x) = \lambda A(x)$. We already know that A is \mathbb{Q}_+ -homogeneous. Let

$$a_\lambda = \{\mu > 0 \mid \lambda + \mu \in \mathbb{Q}_+\}.$$

Thus we have

$$(11) \quad A(\lambda x) + A(\mu x) = A((\lambda + \mu)x) = (\lambda + \mu)A(x) = \lambda A(x) + \mu A(x)$$

for any $\mu \in a_\lambda$, and we need to pass to the limit $\mu \rightarrow 0$ in this relation. The right hand part of (11) converges to $\lambda A(x)$ because \mathcal{V} is monoid. By the assumptions of the proposition $\nu \mapsto A(x\nu)$ is bounded at some $\alpha \geq 0$. Suppose that U is an arbitrary neighbourhood of 0; we need to show that $A(\mu x) \in U$ for all sufficiently small $\mu \in a_\lambda$. By the assumptions there exist some $m > 0$ and $\varepsilon > 0$ satisfying $A(\nu x) \in mU$ for $|\nu - \alpha| < \varepsilon$. For every $\mu > 0$ there exists some $\beta \in \mathbb{Q}_+$ so that $|\beta\mu - \alpha| < \varepsilon$, and therefore $A(\beta\mu x) \in mU$, so $A(\mu x) \in \frac{m}{\beta}U$ by the \mathbb{Q}_+ -homogeneity of A . It is easy to see that we can make $\frac{m}{\beta} \leq 1$ for all sufficiently small μ (for $\alpha > 0$ it is always the case, and in the case $\alpha = 0$ we can take β arbitrarily large by making μ arbitrarily small), which concludes the proof of Proposition 39.

Proposition 40. *Suppose that under the assumptions of Theorem 2 \mathcal{V} is a monoid semilinear space equipped with a Hausdorff topology such that the semilinear operations in \mathcal{V} are continuous and for every $x \in \mathcal{S}$ map $\nu \mapsto T(\nu x)$ is bounded at some $\alpha \geq 0$. Suppose also that the order \leq in \mathcal{V} is continuous in the following sense: if $a_n \leq b_n$ for some $a_n, b_n \in \mathcal{V}$ and b_n converges to 0 then a_n also converges to 0. Then any additive submap $A \leq T$ of T is linear. In particular, the map S defined by (1) is linear.*

Indeed, the proof of Proposition 39 above shows that it is sufficient to establish that $A(\mu x) \rightarrow 0$ as $a_\lambda \ni \mu_j \rightarrow 0$ for any $x \in \mathcal{S}$ and $\lambda > 0$. Since $A \leq T$, by the continuity of the order we can replace A by T . As before, let U be an arbitrary neighbourhood of 0; we need to show that $T(\mu x) \in U$ for all sufficiently small $\mu \in a_\lambda$. By the hypothesis there exist $m > 0$ and $\varepsilon > 0$ such that $T(\nu x) \in mU$ for $|\nu - \alpha| < \varepsilon$. Since a_λ is dense in \mathbb{R}_+ , for every $\delta_j > 0$ and $M_j > 0$ there exists some $\mu_j \in a_\lambda$ and an integer $n > M_j$ satisfying $|n\mu - \alpha| \leq \varepsilon$, and therefore $T(n\mu x) \in mU$. Note that $nT(\mu x) \leq T(n\mu x)$ by the superadditivity of T , so $T(\mu x) \in \frac{m}{n}U$. Since n can be chosen arbitrarily large we can assume that $\frac{m}{n} \leq 1$, and therefore $T(\mu x) \in U$ as claimed. The proof of Proposition 40 is complete.

14. APPLICATION TO INVERSION OF NON-INJECTIVE OPERATORS

The classical Bounded Inverse Theorem (also known as Inverse Mapping Theorem) states that any bounded linear operator $T : X \rightarrow Y$ acting between a couple of Banach spaces has a bounded inverse T^{-1} if T is bounded and bijective (i. e. T is *onto*, that is $T(X) = Y$, and T is injective, that is $Tx = 0$ implies $x = 0$). Its applications include situations when one has a problem in the form of a linear equation $Tx = y$ with respect to x that has unique solutions and question arises whether the solutions of this equation depend continuously on y .

Sometimes, however, solutions are not unique, and thus the Bounded Inverse Theorem is not directly applicable. In this case, i. e. under the same assumptions except injectivity of T , Open Mapping Theorem (also known as the Banach-Schauder theorem) implies that there still is a constant C such that for any $y \in Y$ there exist some $x \in X$ satisfying $Tx = y$ and $\|x\|_X \leq C\|y\|_Y$. Can we find a bounded right inverse for T , i. e. a bounded operator T^{-1} solving $Tx = y$ with respect to y , that is $TT^{-1} = I_Y$ where I_Y is the identity operator in Y ? Incidentally, in the dual terms, i. e. by passing to adjoints, this becomes a question about the existence of a bounded left inverse for T^* if T^* is injective. In certain cases results of this sort are readily available and widely used. Let us examine some of them to look at typical difficulties that may arise. Since T is continuous, the kernel $\mathcal{N}(T) = \{x \in X \mid Tx = 0\}$ of T is closed in X , so the quotient $X/\mathcal{N}(T)$ is also a Banach space, and thus we can apply Bounded Inverse Theorem to obtain a bounded inverse T_*^{-1} for the quotient mapping $T_* : X/\mathcal{N}(T) \rightarrow Y$ defined by $T_*([x]) = T(x)$ for $[x] = \{x + \mathcal{N}(T)\} \in X/\mathcal{N}(T)$ (for more on quotient mappings see, e. g., [10]). If $\mathcal{N}(T)$ is a complemented subspace of X , composition of the projection P onto the complement of $\mathcal{N}(T)$ in X with T_*^{-1} gives a right inverse T^{-1} for T . In particular, if X is a Hilbert space, we can take the orthogonal projection P , which would make the solution of $Tx = y$ given by the resulting operator $T^{-1} = PT_*^{-1}$ have minimal norm in X and thus unique; this technique is known as the least squares method. But what if the kernel $\mathcal{N}(T)$ is not complemented in X ? If $Y = L_1$, it is easy to see that the set-valued map arising from application of the Open Mapping Theorem is actually superlinear on the cone of nonnegative functions $[L_1]_+$, so under certain conditions it is possible to apply Theorem 11 or Theorem 38 to this map.

Theorem 41. *Suppose that X and Y are Banach spaces, Y has a generating cone $\mathcal{S} \subset Y$ satisfying the Riesz decomposition property and the norm $\|\cdot\|_Y$ is linear on \mathcal{S} . Let $T : X \rightarrow Y$ be a continuous linear surjection. Suppose also that at least one of the following conditions are satisfied.*

- (1) X is a dual space, i. e. $X = Z^*$ for some Banach space Z .

- (2) X is a Banach ideal lattice of measurable functions on measurable space \mathbb{Z} satisfying the Fatou property and the kernel $\mathcal{N}(T)$ of T is closed in measure.
- (3) X is a Banach ideal lattice of measurable functions on a σ -finite measurable space satisfying the Fatou property, $Y = L_1$ on a separable measurable space, $\mathcal{S} = [L_1]_+$ and the kernel $\mathcal{N}(T)$ of T is closed in measure.
- (4) $Y = l^1$ and $\mathcal{S} = [l^1]_+$.

Then T admits a bounded right inverse T^{-1} . In particular, T^{-1} embeds Y into X as a subspace, and if Y is not reflexive then X is not reflexive. Also, let z be a point in X satisfying $Tz \in \mathcal{S}$. The following conditions are equivalent.

- (1) T admits a bounded right inverse T^{-1} that preserves the point z , i. e. $T^{-1}Tz = z$.
- (2) For any finite decomposition $y = \sum_{j=1}^N y_j$ of $y = Tz$ into some $y_j \in \mathcal{S}$ there exists a decomposition $z = \sum_{j=1}^N z_j$ of z satisfying $\|z_j\|_X \leq c\|y_j\|_Y$ and $Tz_j = y_j$, with some constant c that does not depend on the decomposition $\{y_j\}$.

Indeed, let us define a set-valued map $\Phi : \mathcal{S} \rightarrow 2^X$ by

$$\Phi(y) = \{x \in X \mid Tx = y, \|x\|_X \leq C\|y\|_Y\}$$

for $y \in \mathcal{S}$. It is easy to see that Φ is superlinear: positive homogeneity is trivial, and for every $f, g \in Y$ conditions $u \in \Phi(f)$, $v \in \Phi(g)$ imply that $\|u + v\|_X \leq \|u\|_X + \|v\|_X \leq C(\|f\|_Y + \|g\|_Y) = C\|f + g\|_Y$ by linearity of the norm $\|\cdot\|_Y$. By the Open Mapping Theorem Φ has nonempty values if C is chosen sufficiently large. Values of Φ are closed in X because T is continuous. Thus, if the first set of assumptions is satisfied then values of Φ are compact in the weak* topology $\sigma(X, Z)$, so Φ acts into a space $\mathcal{V}(\mathcal{W})$ defined in Proposition 8 where we set $\mathcal{W} = X$ with the weak* topology $\sigma(X, Z)$. If the second or third set of assumptions is satisfied then values of Φ are bounded and closed in measure, so Φ takes values in the space $\mathcal{V}(X)$ defined in Proposition 26. Under assumptions 1 and 2 map Φ satisfies conditions of Theorem 11 by either Proposition 8 or Proposition 26, so there exists a linear selection φ of Φ . It is easy to see that map Φ has closed graph and thus it is upper semicontinuous by Proposition 3, so under the third set of assumptions Φ satisfies the conditions of Theorem 38. If the fourth set of assumptions is satisfied then we can construct a suitable linear selection in an obvious way: let e_j be the coordinate vectors in l^1 , choose $z_j \in \Phi(e_j)$, and let

$$(12) \quad \varphi(\{b_j e_j\}) = \sum_j b_j z_j, \quad b_j \geq 0.$$

Thus under any of the conditions 1–4 map Φ has a linear selection φ . Since the cone \mathcal{S} is generating for the space Y , we can define a map $T^{-1} : Y \rightarrow X$ as the linear extension of φ by Proposition 1. It is easy to see that T^{-1} is a right inverse of T with norm at most C . The rest of the statement 41 is an interpretation of the conditions of Theorem 11 or Theorem 38 for the map Φ . Suppose that T admits a bounded right inverse T^{-1} with norm c . Then for any finite decomposition $y = \sum_{j=1}^N y_j$ of $y = Tz \in \mathcal{S}$ into some elements of \mathcal{S} elements $z_j = T^{-1}y_j$ form a decomposition $z = \sum_{j=1}^N z_j$ of z satisfying $\|z_j\|_X \leq \|T^{-1}\|_{Y \rightarrow X} \|y_j\|_Y \leq c \|y_j\|_Y$. Conversely, if under any of assumptions 1–3 condition 2 is satisfied and we take constant C in map Φ to be at least c , then by Theorem 11 or Theorem 38 there exists a linear selection φ of Φ satisfying $\varphi(Tz) = z$, and therefore $T^{-1}Tz = z$ for the map T^{-1} . Assumptions 4 involve a bit of manual labor. Let $y = \sum_{j=1}^{\infty} a_j e_j$. We apply condition 2 to a sequence of decompositions $y = \sum_{j=1}^N a_j e_j + \sum_{j>N} a_j e_j$ to obtain $z_{N,j} \in X$ satisfying $Tz_{N,j} = a_j e_j$ for $1 \leq j \leq N$, $Tz_{N,N+1} = \sum_{j>N} a_j e_j$ and $\sum_{j=1}^{N+1} z_{N,j} = z$. It is evident that we can take $z_j = z_{j,j} = z_{N,j}$ for all j and thus $y_j = Tz_{j,j}$ and $\sum_j z_{j,j} = z$, so the selection (12) satisfies the necessary properties. The proof of Theorem 41 is complete.

We mention that there may not be a continuous right inverse for operator T if conditions of Theorem 41 are not satisfied. An interesting example of this is a theorem of Peetre [28] stating that the trace operator Tr defined on the Sobolev space $L_1^{(1)}(\mathbb{R}_+^{n+1})$ acting from the upper half-space \mathbb{R}_+^{n+1} to $L_1(\mathbb{R}^n)$, despite being onto (see, e. g., [29, Lemma 19]), admits no bounded right inverse; see [29, Theorem 20].

15. APPLICATION TO APPROXIMATION OF LINEAR SPACES

In this section we mention how selection theorems for superlinear maps could be applied to certain questions related to approximation. There is a huge amount of literature dedicated to various problems of approximation of functions. Although I was not successful in finding any concrete useful applications for what follows in this section, the general principle seems to be interesting enough to include in this survey.

Let X be a normed vector space and Y its subspace. The value

$$(13) \quad E_Y(f) = \inf_{P \in Y} \|P - f\|_X$$

determines how well a given function $f \in X$ can be approximated by a function from Y . There usually are two main questions associated with approximation: how the value $E_Y(f)$ can be estimated for various subspaces Z of X , and how the best (or, failing that, merely just good enough) approximating function can be obtained. An answer to the

first question usually has the form

$$(14) \quad E_Y(f) \leq C \|f\|_Z$$

with some constant C and a seminorm $\|\cdot\|_Z$ (or something like a seminorm, e. g. a modulus of continuity). Estimates of the form (14) are usually called *Jackson-type inequalities*. A classical example is the space of continuous functions $X = C(\mathbb{T})$ on the unit circle \mathbb{T} with the usual supremum norm, the space Y_n of trigonometric polynomials on \mathbb{T} of degree at most $n - 1$ and the space $Z = C^{(r)}(\mathbb{T})$ of r times continuously differentiable functions with seminorm $\|f\|_{C^{(r)}} = \sup_{z \in \mathbb{T}} |f^{(r)}(z)|$. In this case (14) is known as the classical Jackson inequality with $C = \frac{C_r}{n^r}$, where C_r (which is known as the Akhiezer–Krein–Favard constant) depends only on r ; see, e. g., [1].

The nature of the approximating transformation $f \mapsto P$ used to obtain inequality (14) is rarely explicitly mentioned in the literature. However, on closer examination, the approximating function P is nearly always expressed as a linear transformation of the function f being approximated, even in cases where one might expect some degree of nonlinearity (for example, if weighted spaces are involved; see, e. g., [25]). Thus a question arises: can the approximation satisfying an estimate (14) be carried out by a linear transformation, i. e. does (14) imply that there exists a linear operator $T : X \rightarrow Y$ satisfying

$$(15) \quad \|f - Tf\|_X \leq C \|f\|_Z$$

for all $f \in X$? Some answers are readily available. Similarly to what was discussed in Section 14, if X is a Banach space and Y is a closed subspace of X , then $E_Y(f)$ is just the norm of f in the quotient space X/Y , and the question is, essentially, whether we can find a bounded projection from $(X/Y) \cap Z$ onto X . If $X = Z$ is a Hilbert space and Y is a closed subspace of X then the orthogonal projection from X onto Y solves this problem. One might consider a question stronger than (15) about the best approximation, i. e. if we can find an operator $T : Z \rightarrow Y$ providing the best (or “almost the best” in some sense) possible approximation in X by Y for every function $f \in Z$. There is a set-valued map $\pi_Y : X \rightarrow \mathcal{C}(Y)$ defined by

$$(16) \quad \pi_Y(f) = \left\{ Q \mid \|Q - f\|_X \leq (1 + \varepsilon) \inf_{P \in Y} \|P - f\|_X \right\}$$

that gives approximating functions P for any $\varepsilon > 0$. As there usually is a function P realizing the infimum in (13) (for example, this is always the case if Y is finite-dimensional), we can take $\varepsilon = 0$ in (16). In this case π_Y is called the *metric projection*. If π_Y admits a linear selection then a linear mapping T satisfying 15 also exists, but apart from the Hilbert space case this happens only rarely. For example, in [8, Theorem 3.5] it was shown for the case $X = L_1$ on a measurable space (Ω, μ) that if Y is a finite-dimensional nontrivial subspace of X

and π_Y admits a linear selection then measure μ has at least one atom. The situation with (15) is much better as it is easy to see that in some cases the corresponding set-valued map is superlinear, so we can apply results of this survey to obtain a linear selection. For simplicity we will state the following theorem for finite-dimensional spaces Y only.

Theorem 42. *Suppose that X is a normed space, $Y, Z \subset X$ are linear sets, Z is equipped with a quasiseminorm with respect to which estimate (14) holds for all $f \in Z$, $\mathcal{S} \subset Z$ is a generating cone for Z that satisfies the Riesz decomposition property and the quasiseminorm $\|\cdot\|_Z$ is 1-concave on \mathcal{S} (i. e. $\|f\|_Z + \|g\|_Z \leq \|f + g\|_Z$ for any $f, g \in \mathcal{S}$). Suppose also that Y is finite-dimensional. Then there exists a linear operator $T : Z \rightarrow Y$ satisfying (15).*

Indeed, since Y is finite-dimensional, the infimum in (14) is always attained. Define a set-valued map $\Phi : \mathcal{S} \rightarrow 2^Y \setminus \{\emptyset\}$ by

$$\Phi(f) = \{P \in Y \mid \|f - P\|_X \leq C\|f\|_Z\}$$

for $f \in \mathcal{S}$. It is evident that Φ is positive homogeneous and from 1-concavity of the norm of Z on \mathcal{S} it follows that for any $f, g \in \mathcal{S}$, $P \in \Phi(f)$ and $Q \in \Phi(g)$ we have

$$\begin{aligned} \|f + g - (P + Q)\|_X &\leq \|f - P\|_X + \|g - Q\|_X \leq \\ &C(\|f\|_Z + \|g\|_Z) \leq C\|f + g\|_Z, \end{aligned}$$

i. e. $P + Q \in \Phi(f + g)$, and therefore Φ is superlinear. As Y is finite-dimensional, it is easy to see that values of Φ are compact in Y , so by Theorem 11 there exists a linear selection φ of Φ . Since \mathcal{S} is generating for Z , we can extend φ to a linear operator T on Z by Proposition 1. Thus T satisfies (15), which concludes the proof of Theorem 42.

Let us give an example how Theorem 42 can be applied to what is sometimes referred to as the Jackson-Favard estimate for suitable functions f on the segment $[-1, 1]$, $1 \leq p \leq \infty$ and $n \geq r$ (see, e. g., [9, Chapter 8, (7.1)]):

$$(17) \quad \inf_{P \in \mathbb{P}_n} \|P - f\|_{L_p} \leq C_r n^{-r} \|\varphi^r f^{(r)}\|_{L_p},$$

where $Y = \mathbb{P}_n$ denotes the space of all polynomials of degree at most n , and $\varphi(x) = \sqrt{1 - x^2}$ for $x \in [-1, 1]$. Note that in this case Theorem 42 does not give any new information (apart from the equality of optimal constants in (17) and (18) below) since a suitable linear operator T is already implicit in the proof of the inequality. In order to apply Theorem 42, we have to restrict ourselves to the case $p = 1$. Then we take $X = L_1$, Z to be the Sobolev space of measurable functions f that are r times differentiable almost everywhere, $f^{(k)}(0) = 0$ for $0 \leq k < r$, and their norm $\|f\|_Z = \|\varphi^r f^{(r)}\|_{L_1}$ is finite, $\mathcal{S} = \{f \in Z \mid f^{(r)} \geq 0 \text{ a. e.}\}$ is isomorphic to $[L_1]_+$ for a weighted space L_1 , so it has the Riesz decomposition property. Then application of Theorem 42 shows that there

exists a linear operator T acting into \mathbb{P}_n that satisfies

$$(18) \quad \|Tf - f\|_{L_1} \leq C_r n^{-r} \|\varphi^r f^{(r)}\|_{L_1}$$

for all suitable f .

16. APPLICATION TO ANALYTIC FUNCTIONS ON POLYDISK

In this section we will show how Theorem 11 or Theorem 34 can be applied to a construction of analytic functions in polydisk \mathbb{D}^n , $n \geq 2$, having prescribed nonnegative radial boundary values almost everywhere. The result of this application detailing a linear construction with suitable control will be used below in Section 17 to obtain a weak version of the corona theorem on polydisk. For basic facts regarding the theory of analytic functions on polydisk see, e. g., [33].

Let $RP(\mathbb{T}^n) \subset M(\mathbb{T}^n)$ be the set of Borel measures on \mathbb{T}^n having finite variation such that convolution of the Poisson kernel with any $\mu \in RP(\mathbb{T}^n)$ is a real part of an analytic function in \mathbb{D}^n . It is well known (see, e. g., [33]) that μ belongs to $RP(\mathbb{T}^n)$ if and only if the spectrum $\text{supp } \hat{\mu}$ of the measure μ belongs to the set

$$\widehat{RP}_n = \mathbb{Z}_+^n \cup (-\mathbb{Z}_+^n) = \{j = (j_1, \dots, j_n) \mid \sigma j \geq 0 \text{ for some } \sigma \in \{-1, 1\}\}.$$

A major difference between the cases of disk ($n = 1$) and polydisk ($n \geq 2$) is this restriction on the spectrum of the real part of an analytic function (in the sense of boundary values). This, among other things, presents considerable challenges when one tries to construct analytic functions in polydisk with controls on the boundary values; a well-known example of this is that there is no factorization in $H_p(\mathbb{D}^n)$, $n \geq 2$. However, there are some weaker results. By [33, Theorem 2.4.2] for any lower semicontinuous $f \in [L_1(\mathbb{T}^n)]_+$ we can find a non-negative singular measure $\sigma_f \in M(\mathbb{T}^n)$ satisfying $f - \sigma_f \in RP(\mathbb{T}^n)$ and $\|\sigma_f\|_{M(\mathbb{T}^n)} = \|f\|_{L_1(\mathbb{T}^n)}$. In other words, for any such f we can find an analytic function F in \mathbb{D}^n such that $\Re F(0) = 0$ and its radial boundary values F^* satisfy $\Re F^* = f$ almost everywhere on \mathbb{T}^n . We extend this result to show that σ_f can be chosen to depend linearly on f , which will be very useful in Section 17 below. First, we need a more precise statement of [33, Theorem 2.4.2].

Let G_m be the subgroup

$$G_m = \left\{ (e^{i\theta_1}, \dots, e^{i\theta_n}) \in \mathbb{T}^n \mid \exp \left(i2^m \sum_{j=1}^n \theta_j \right) = 1 \right\}$$

of the group \mathbb{T}^n with respect to pointwise multiplication for any integer $m \geq 0$. It is easy to see that G_m is continuous and compact with respect to the topology induced from \mathbb{T}^n . Let μ_m be the Haar measure of the group G_m . Observe that μ_m is, up to a normalization factor of order 2^{-m} , the $(n-1)$ -dimensional Lebesgue measure on the diagonal sections

$\left\{e^{i\theta} \mid \sum_{j=1}^n \theta_j = 2\pi \frac{k}{2^m}\right\}$ of the torus \mathbb{T}^n , $k \in \mathbb{Z}$. Let $\mu = \sum_{k \geq 0} \mu_k$; it is easy to see that μ is a σ -finite singular Borel measure on \mathbb{T}^n . Let $G = \bigcup_{m \geq 0} G_m$ with Borel σ -algebra induced from \mathbb{T}^n . For convenience, let $G_{-1} = \emptyset$. Observe that $\chi_{G_s} \mu_m = 2^{s-m} \chi_{G_s} \mu_s$ for all $m \geq s$. It is easy to see, using this scaling condition, that

$$\chi_{G_s \setminus G_{s-1}} \mu = \sum_{k \geq s} \chi_{G_s \setminus G_{s-1}} \mu_k = \sum_{k \geq s} \chi_{G_s \setminus G_{s-1}} 2^{s-k} \mu_s = 2 \chi_{G_s \setminus G_{s-1}} \mu_s$$

for all $s \geq 0$. Therefore, again using that $\mu_m = 2^{s-m} \mu_s$ on G_s for $s \leq m$,

$$\begin{aligned} \mu_m &= \sum_{s=0}^m \chi_{G_s \setminus G_{s-1}} \mu_m = \sum_{s=0}^m \chi_{G_s \setminus G_{s-1}} 2^{s-m} \mu_s = \\ &= \sum_{s=0}^m \chi_{G_s \setminus G_{s-1}} 2^{s-m-1} \mu = w_m d\mu \end{aligned}$$

for all $m \geq 0$ with $w_m = \sum_{s=0}^m \chi_{G_s \setminus G_{s-1}} 2^{s-m-1}$.

Proposition 43. *Let f be a lower semicontinuous function from $L_1(\mathbb{T}^n)$ which is positive everywhere on \mathbb{T}^n . Then there exists a μ -measurable nonnegative function ρ satisfying the following conditions: $\rho \leq \frac{1}{2}f$ μ -almost everywhere, $\rho \in L_1(d\mu)$, $f d\mu - \rho d\mu \in RP(\mathbb{T}^n)$ and*

$$\|\rho d\mu\|_{M(\mathbb{T}^n)} = \|\rho\|_{L_1(d\mu)} = \|f\|_{L_1(\mathbb{T}^n)}.$$

The proof follows the argument of [33, Theorem 2.4.2]. As f is positive and lower semicontinuous, it is easy to construct a decomposition $f = \sum_{m \geq 1} f_m$ into nonnegative trigonometric polynomials f_m of degree at most $2^m - 1$, $m \in \mathbb{N}$. Let $\rho = \sum_{m \geq 1} w_m f_m$. Thus $\rho d\mu = \sum_{m \geq 1} f_m d\mu_m = \sum_{m \geq 1} \sigma_m$ with $\sigma_m = f_m d\mu_m$. Representing

$$f_m(\zeta) = \sum_{|l| < 2^m} \hat{f}_m(l) \zeta^l$$

with the standard multi-index notation, we see that

$$\begin{aligned} (19) \quad \hat{\sigma}_m(k) &= \int_{\mathbb{T}^n} e^{-i\theta \cdot k} \sum_{|l| < 2^m} f_m(\theta) d\mu_m(\theta) = \\ &= \int_{\mathbb{T}^n} e^{-i\theta \cdot k} \sum_{|l| < 2^m} \hat{f}_m(l) e^{i\theta \cdot l} d\mu_m(\theta) = \\ &= \sum_{|l| < 2^m} \hat{f}_m(l) \int_{\mathbb{T}^n} e^{i(l-k) \cdot \theta} d\mu_m(\theta) = \hat{f}_m(k) \end{aligned}$$

for $k \in [\mathbb{Z}^n \setminus (\mathbb{Z}_+^n \cup (-\mathbb{Z}_+^n))] \cup \{0\}$, since by the orthogonality of the characters of the group G_m (or a direct computation since we know what the measure μ_n is) the value of $\int_{\mathbb{T}^n} e^{i(l-k) \cdot \theta} d\mu_m(\theta)$ is 1 for

$k - l = 2^m s \mathbf{1}$ with some $s \in \mathbb{Z}$ where $\mathbf{1} = (1, \dots, 1)$, and 0 otherwise. Furthermore,

$$\begin{aligned} \|\rho\|_{L_1(\mu)} &= \sum_{m \geq 1} \|w_m f_m\|_{L_1(\mu)} = \sum_{m \geq 1} \int f_m d\mu_m = \\ &= \sum_m \hat{\sigma}_m(0) = \sum_m \hat{f}_m(0) = \sum_m \int f_m = \|f\|_{L_1} \end{aligned}$$

and

$$\rho = \sum_{m \geq 1} w_m f_m \leq \frac{1}{2} \sum_{m \geq 1} f_m = \frac{1}{2} f$$

μ -almost everywhere because $w \leq \frac{1}{2}$ μ -almost everywhere. Finally, (19) means that

$$\hat{f}(k) - \hat{\sigma}(k) = \sum_m \left(\hat{f}_m(k) - \hat{\sigma}_m(k) \right) = 0$$

for $k \notin \mathbb{Z}_+^n \cup (-\mathbb{Z}_+^n)$, so $f dm - \sigma \in RP(\mathbb{T}^n)$ as claimed. The proof of Proposition 43 is complete.

Let C_{0+} be the cone of continuous functions on \mathbb{T}^n that are positive everywhere on \mathbb{T}^n . It is easy to see that C_{0+} is a lattice and therefore it has the Riesz decomposition property. We are now ready to prove the main result of this section.

Theorem 44. *For every $n \in \mathbb{N}$, $n \geq 2$, there exists a bounded with norm 1 linear mapping $Y : L_1(\mathbb{T}^n, dm) \rightarrow L_1(G, d\mu)$ satisfying*

$$f dm - (Yf)d\mu \in RP(\mathbb{T}^n)$$

for all $f \in L_1(\mathbb{T}^n, dm)$. Moreover, operator Y is positive, i. e. $f \geq 0$ almost everywhere implies that $Yf \geq 0$ μ -almost everywhere. If $f \in C_{0+}$ then $Yf \leq \frac{1}{2}f$ on \mathbb{T}^n .

Define a set-valued map $\Phi : C_{0+} \rightarrow 2^{L_1(\mu)}$ by

$$\begin{aligned} \Phi(h) = \left\{ \rho \in L_1(\mu) \mid \rho \geq 0 \text{ a. e., } h dm - \rho d\mu \in RP(\mathbb{T}^n), \right. \\ \left. \rho \leq \frac{1}{2}h \text{ a. e., } \|\rho\|_{L_1(\mu)} \leq \|h\|_{L_1} \right\} \end{aligned}$$

for $h \in C_{0+}$. By Proposition 43 map Φ takes nonempty values. Let us verify that values of Φ are compact in the weak topology of $L_1(\mu)$. Observe that for every $h \in C_{0+}$ the set $\Phi(h)$ is uniformly integrable because it is dominated by a bounded function $\frac{1}{2}h$, which by the Dunford-Pettis theorem means that $\Phi(h)$ is relatively compact in the weak topology of $L_1(\mu)$. Thus it suffices to show that $\Phi(h)$ is closed in this topology for every $h \in C_{0+}$. Fix $h \in C_{0+}$ and let $\rho_\alpha \in \Phi(h)$ be a net converging weakly to some $\rho \in L_1(\mu)$. Since by the weak convergence for every measurable set $E \subset G$ we have $\int_E \rho = \lim_\alpha \int_E \rho_\alpha \leq \int_E \frac{1}{2}h$, it

follows at once that $\rho \leq \frac{1}{2}h$. Since functions $\zeta \rightarrow e^{in\zeta}$ are bounded and continuous, the Fourier coefficients of the measure $h dm - \rho_\alpha d\mu$ converge to the Fourier coefficients of the measure $h dm - \rho_\alpha d\mu$. Therefore $h dm - \rho d\mu \in RP(\mathbb{T}^n)$. Finally, $\|\rho\|_{L_1(\mu)} \leq \liminf_\alpha \|\rho_\alpha\|_{L_1(\mu)} \leq \|h\|_{L_1}$ by the Fatou theorem. We have verified that $\Phi(h)$ is closed and thus compact in the weak topology of $L_1(\mu)$. It is easy to see that $\Phi(h)$ is convex for every $h \in C_{0+}$. In short, Φ takes values in the set $\mathcal{V}(L_1(\mu))$ of all nonempty compact convex sets with respect to the weak topology of $L_1(\mu)$ as defined in Proposition 8. It is easy to see that Φ is a superlinear map, so by Theorem 11 combined with Proposition 8 there exists a linear selection Y of Φ that satisfies the claims of Theorem 44 on the set C_{0+} .

Let us now extend Y onto the cone $C_+(\mathbb{T}^n)$ of all nonnegative continuous functions on \mathbb{T}^n . Set $Yf = Y(f + \delta) - Y(\delta)$ for any $\delta > 0$ and all $f \in C_+(\mathbb{T}^n)$. This is an extension since for $f \in C_{0+}$ we have $Y(f + \delta) - Y(\delta) = Y(f) + Y(\delta) - Y(\delta) = Y(f)$ due to the additivity of Y . It is easy to see that this definition does not depend on the value $\delta > 0$: if $\delta_2 > \delta_1$ then

$$\begin{aligned} Y(f + \delta_2) - Y(\delta_2) &= Y(f + \delta_1) + Y(\delta_2 - \delta_1) - Y(\delta_2) = \\ &= Y(f + \delta_1) + Y(\delta_2 - \delta_1) - (Y(\delta_2 - \delta_1) + Y(\delta_1)) = \\ &= Y(f + \delta_1) - Y(\delta_1). \end{aligned}$$

Map Y extended this way remains linear: positive homogeneity is conserved because

$$Y(\lambda f) = Y(\lambda f + \lambda \delta) - Y(\lambda \delta) = \lambda[Y(f + \delta) - Y(\delta)] = \lambda Y(f)$$

for all $f \in C_+(\mathbb{T}^n)$ and $\lambda > 0$, and additivity is conserved because

$$\begin{aligned} Y(f + g) &= Y(f + g + 2\delta) - Y(2\delta) = \\ &= [Y(f + \delta) - Y(\delta)] + [Y(g + \delta) - Y(\delta)] = Y(f) + Y(g) \end{aligned}$$

for all $f, g \in C_+(\mathbb{T}^n)$. This extension is still bounded with norm 1 because

$$\begin{aligned} \|Y(f)\|_{L_1(\mu)} &\leq \inf_{\delta > 0} \|Y(f + \delta) - Y(\delta)\|_{L_1(\mu)} \leq \\ &\leq \inf_{\delta > 0} (\|Y(f + \delta)\|_{L_1(\mu)} + \|Y(\delta)\|_{L_1(\mu)}) \leq \inf_{\delta > 0} (\|f + \delta\|_{L_1} + \|\delta\|_{L_1}) \leq \\ &\leq \inf_{\delta > 0} (\|f\|_{L_1} + 2\|\delta\|_{L_1}) = \|f\|_{L_1} \end{aligned}$$

for all $f \in C_+(\mathbb{T}^n)$. Since $C_+(\mathbb{T}^n)$ is a generating cone for $C(\mathbb{T}^n)$, which is a lattice with norm $\|\cdot\|_{L_1}$, we can apply Proposition 1 to extend Y to a positive operator $Y : C(\mathbb{T}^n) \rightarrow L_1(\mu)$ with norm 1. Then, because $C(\mathbb{T}^n)$ is dense in L_1 we can extend Y to L_1 by continuity with preservation of norm. Condition $f dm - (Yf)d\mu \in RP(\mathbb{T}^n)$ for all $f \in C(\mathbb{T}^n)$ is preserved throughout the extension because the Fourier

coefficients $\widehat{\psi d\mu}(k) = \int_{\mathbb{T}^n} e^{-ik\zeta} \psi d\mu$, $k \in \mathbb{Z}^n$ are linear and continuous in $L_1(G, d\mu)$. The proof of Theorem 44 is complete.

17. APPLICATION TO THE CORONA PROBLEM

In this section we will show how Theorem 44 can be applied to partially extend the result obtained in [21] (see also [36] for a simpler version) to the case of polydisk. See, e. g., [42], [11] and references contained therein for detailed background on the corona problem from the analytic point of view.

One frequently used reformulation of the corona problem in a standard setting is stated roughly as follows. Suppose that $G \subset \mathbb{C}^m$ is a domain in \mathbb{C}^m , $m \geq 1$, and we are given a finite collection of bounded analytic functions $f_j \in H_\infty(G)$ satisfying $\sup_{z \in G} \sum_j |f_j(z)| > 0$; can we find some bounded analytic functions $g_j \in H_\infty(G)$ solving the equation $\sum_j f_j g_j = 1$? As of the time of this writing, for $m = 1$ and $G = \mathbb{D}$ this question has been settled for almost five decades, but *no* domains $G \subset \mathbb{C}^m$, $m \geq 2$ are known for which the corona problem has positive solution. On the other hand, there are examples of smooth pseudoconvex (but not strictly pseudoconvex) domains for in which the corona problem does not have a solution in general; see [13] and [38]. However, there are a fairly large number of positive results for fairly regular domains, e. g. in polydisk \mathbb{D}^m or strictly pseudoconvex domains like the unit ball \mathbb{B}_m of \mathbb{C}^n , if we replace the requirement of uniform boundedness of g_j with some weaker conditions that still provide quite a lot of control over the behavior of g_j , for example $g_j \in \text{BMO}$ (in the sense of boundary values), or even $e^{\left[(|g_j|/\lambda)^{\frac{1}{2m-1}} \right]} \in L_1$ for some $\lambda > 0$ (also in the sense of boundary values); see, e. g., [43].

Another related question is related to the *Toeplitz Corona Theorem*, which roughly means that we replace 1 in the right part of the equation $\sum_j f_j g_j = 1$ with an arbitrary function $h \in X$, where X is some normed space of analytic functions, and try to solve $\sum_j f_j g_j = h$ with respect to $g_j \in X$ with a suitable estimate on the norms of g_j in X . This question also appears to be more yielding with a significant number of results for various domains and norms, most notably H_p for $p < \infty$ and, in particular, H_2 , which corresponds to the original Toeplitz Corona Theorem. The case $X = H_p(\mathbb{D}^n)$ was first solved in [24], [23], and there are a number of results with improved estimates and for other domains in \mathbb{C}^n ; see, e. g., [42], [44], [2], [3] and references contained therein. Let us introduce a latest result for the polydisk obtained in [42]. We use the notation $l_N^2 = \{a \in l^2 \mid \text{supp } a \subset \{k \mid 1 \leq k \leq N\}\}$ for $N \in \mathbb{N}$.

Theorem 45. [42] *Let $1 \leq p < \infty$. Suppose that $m > 1$, $k \in \mathbb{N}$, $N \in \mathbb{N}$ and $F \in H_\infty(\mathbb{D}^m; l_N^2 \rightarrow l_k^2)$ satisfies $\delta^2 I \leq FF^* \leq I$ and $0 < \delta^2 < \frac{1}{e}$.*

Let $f \in H_p(\mathbb{D}^m, l_k^2)$. Then there exists some function $g \in H_p(\mathbb{D}^m, l_N^2)$ satisfying $Fg = f$ and $\|g\|_{L_p(l_N^2)} \leq C\|f\|_{H_\infty(l_k^2)}$ with a constant C independent of f and N .

The classical Toeplitz corona theorem corresponds to $p = 2$ and $m = 1$. This approach seems promising because there is some hope in extracting from it some useful information for the case $p = \infty$. It is well known that for $m = 1$ cases $p = 2$ and $p = \infty$ are equivalent with equivalence of constants; see, e. g., [27, Appendix 3]. Recently in [21] this equivalence was extended to a wide variety of Hardy-type spaces, in particular to all H_p , $1 \leq p \leq \infty$ for $m = 1$. Unfortunately, the method used in [21] depends heavily on factorization, which makes it unsuitable for $m > 1$. However, if we replace factorization with a surrogate provided by Theorem 44 in Section 16, it is possible to obtain from Theorem 45 a weaker result using these methods.

Theorem 46. Suppose that $m > 1$, $k \in \mathbb{N}$, $N \in \mathbb{N}$ and

$$F \in H_\infty(\mathbb{D}^m; l_N^2 \rightarrow l_k^2)$$

satisfies $\delta^2 I \leq FF^* \leq I$ and $0 < \delta^2 < \frac{1}{e}$. Let $f \in H_\infty(\mathbb{D}^m, l_k^2)$. Then there exists some function g analytic in \mathbb{D}^m such that $Fg = f$ and the radial boundary values g^* of g satisfy $\|g^*\|_{L_\infty(l_N^2)} \leq C\|f\|_{H_\infty(l_k^2)}$ with a constant C independent of f and N .

Note that analytic functions with radial limits bounded a. e. may still have very nasty behavior. As it appears from the construction below, the solution may behave like a ratio of two functions from H_p not implicitly known, a far worse control inside the polydisk than what is typically implied the results mentioned before since the denominator may take arbitrarily small values inside the polydisk. Nevertheless, solutions that are “bounded in measure”, i. e. with radial values converging in measure to a bounded function, appear to be new and may still be useful in some applications.

We now begin the proof of Theorem 46. Denote by B the closed unit ball of $H_2(l^2)$ with the weak topology. The space B is metrizable, and convergence of a sequence $f_n = \{f_j^{(n)}\}$ is equivalent to pointwise convergence of functions $f_j^{(n)}$ on the polydisk \mathbb{D}^m for all j . Fix any $\delta > 1$. Let P_r , $0 \leq r < 1$, be the Poisson integral in \mathbb{D}^m . We denote by $P_z\nu$ and $Q_z\nu$ convolutions of a measure ν with the Poisson kernel and the conjugate Poisson kernel for a point $z \in \mathbb{D}^m$ respectively. The modulus sign $|\cdot|$, as usual, denotes the norm in the sequence space l^2 ; in this argument all such sequences are finite. Let

$$f_g(z) = \frac{1}{C(1+\delta)} \exp[(P_z + iQ_z)(\log(\delta + |P_r g|) dm - [Y \log(\delta + |P_r g|)] d\mu)]$$

for $g \in B$ and $z \in \mathbb{D}^m$, where C is the constant in Theorem 45 applied to functions F and $f = g$ with $p = 2$, singular Borel measure μ on \mathbb{T}^n is the one defined in Section 16 and Y is a mapping defined by Theorem 44.

Note that every function f_g is analytic, has no zeroes in \mathbb{D}^m and satisfies estimates $|f_g| \geq \frac{\delta}{C(1+\delta)}$ and $\|f_g\|_{H_2} \leq \frac{1}{C}$. Now define a set-valued map $\Phi : B \rightarrow 2^B$ by

$$\Phi(g) = \{h \in B \mid Fh = f_g f\}.$$

By Theorem 45, for all $g \in B$ the values $\Phi(g)$ are nonempty. It is easy to see that $\Phi(g)$ is convex and closed in B . Let us verify that the graph of Φ is closed. Indeed, let $g_n \in B$, $h_n \in \Phi(g_n)$ and suppose that for some $g, h \in B$ sequences g_n and h_n converge in B to some g and h respectively. Then f_{g_n} converges to f_g pointwise in \mathbb{D}^m , h_n also converges to h pointwise in \mathbb{D}^m , so we can pass to a limit in the relation $Fh_n = f_{g_n} f$. Thus B and Φ satisfy the conditions of the Ky Fan–Kakutani fixed point theorem.

Theorem (Ky Fan–Kakutani [12]). *Suppose that K is a compact set in a locally convex linear topological space. Let Φ be a mapping from K to the set of nonempty compact convex subsets of K . If the graph $\text{gr } \Phi = \{(x, y) \in K \times K \mid y \in \Phi(x)\}$ of Φ is closed in $K \times K$ then Φ has a fixed point, i. e. $x \in \Phi(x)$ for some $x \in K$.*

It follows that for every r there exists some $g_r \in B$ satisfying $g_r \in \Phi(g_r)$, i. e.

$$(20) \quad Fg_r = f_{g_r} f.$$

Since B is compact and metrizable, there exists an increasing sequence $r_n \rightarrow 1$ such that g_{r_n} converges to some $g \in B$ in B and therefore $g_{r_n} \rightarrow g$ pointwise in \mathbb{D}^m . For the same reasons, passing to a subsequence, we can assume that $f_{g_{r_n}}$ converges to some analytic function G uniformly on compact sets in \mathbb{D}^m . The sequence $a_n = \log(\delta + |P_{r_n} g_{r_n}|)$ is bounded in L_2 ; therefore, passing to a subsequence, we can assume that a_n converges weakly in L_2 to some function a . Since $a_n \geq \log \delta$ almost everywhere, it is easy to conclude from the weak convergence that $a \geq \log \delta$ almost everywhere. Now observe that $b \mapsto (P_z + iQ_z)(b \, d\mu - [Yb]d\mu)$ is a continuous linear functional on L_2 (here we make use of the continuity of $Y : L_1(\mathbb{T}^n, d\mu) \rightarrow L_1(G, d\mu)$ established in Theorem 44 and continuity of the inclusion $L_2 \subset L_1$). Therefore analytic functions $(P_z + iQ_z)(a_n d\mu - [Ya_n]d\mu)$ converge to $(P_z + iQ_z)(a \, d\mu - [Ya]d\mu)$ pointwise in \mathbb{D}^m , and so functions $f_{g_{r_n}}$ converge pointwise in \mathbb{D}^m to

$$(21) \quad G(z) = \frac{1}{C(1+\delta)} \exp[(P_z + iQ_z)(a \, d\mu - [Ya]d\mu)].$$

Surely, (20) implies that

$$(22) \quad Fg = Gf$$

on \mathbb{D}^m .

Passing to a subsequence once again, we can assume that the sequence $h_{r_n} = P_{r_n}g_{r_n}$ converges weakly to some $h \in B$. Then for every $0 < r < 1$ we have

$$P_r h = \lim_{n \rightarrow \infty} P_r h_{r_n} = \lim_{n \rightarrow \infty} P_r g_{r_n} + \lim_{n \rightarrow \infty} (P_r P_{r_n} - P_r) g_{r_n} = P_r g$$

almost everywhere because $\lim_{n \rightarrow \infty} \|P_r(P_{r_n} - I)\|_{L_1 \rightarrow C(\mathbb{T}^m)} \rightarrow 0$. Therefore $h = g$ everywhere on \mathbb{D}^m and almost everywhere on \mathbb{T}^m .

Finally, since functions $\log |h_{r_n}|$ are plurisubharmonic and converge weakly to h ,

$$(23) \quad |g(z)| = \lim_n |h_{r_n}(z)| \leq \lim_n \exp(P_z[\log |h_{r_n}|]) \leq \\ \lim_n \exp(P_z[\log(\delta + |h_{r_n}|)]) = \lim_n \exp(P_z a_n) = \exp(P_z a) = \\ C(1 + \delta)|G(z)| \exp(P_z[(Ya)d\mu])$$

for all $z \in \mathbb{D}^m$ using (21). Since measure $(Ya)d\mu$ is singular, passing in (23) to radial boundary values yields

$$|g^*| \leq C(1 + \delta)|G^*|$$

almost everywhere. By the construction G has no zeroes in \mathbb{D}^m , so function $b = \frac{g}{G}$ satisfies $Fb = f$ by (22) with the claimed estimates on b^* . The proof of Theorem 46 is complete.

18. CONCLUDING REMARKS

Semilinear spaces introduced in Section 1 have been studied to some extent and occasionally appear in various contexts; see, e. g., [46], [45].

Axiom 3 in our definition of a semilinear topological space given in Section 3 and used throughout the paper is actually a trick that allows us to entirely bypass issues of lower semicontinuity and work exclusively with upper semicontinuity where it is needed. Under appropriate restrictions upper semicontinuity implies the usual continuity of set-valued maps defined by a Hausdorff topology (i. e. the topology corresponding to the Hausdorff metric); see, e. g., [26] and references contained therein.

The case of finite \mathcal{D} in Theorem 9 of Section 4 roughly corresponds to the case of maps taking values in bounded closed convex sets of a finite-dimensional space $\mathcal{W} = \mathbb{R}^n$ that are automatically compact, raising a question whether a suitable target space $\mathcal{V}(\mathcal{W})$ always has compact type if it admits a finite exhaustive set \mathcal{D} of linear functionals that are defining for and consistent with \mathcal{W} . Nevertheless, the point is that one way or another the compact type of \mathcal{V} is not explicitly needed in the proof if the set \mathcal{D} is finite.

It might seem that the method of obtaining linear selections of linear maps described in Section 5, not entirely dissimilar to other commonly used techniques (see, e. g., [4, Chapter 9]), may have further applications to other selection problems. However, it is easy to construct an example of a nice continuous set-valued map having discontinuous tomographical selections. Let $A : \mathbb{R} \rightarrow \mathcal{C}(\mathbb{R}^2)$ be defined by $A(x) = [(0, 0), (x, 1)]$, i. e. $A(x)$ is the segment connecting the origin with point $(x, 1)$, and $\mathcal{D} = \{f_x, f_y\}$ be the coordinate functionals $f_x((x, y)) = x$ and $f_y((x, y)) = y$ in this order. Then the corresponding tomographical selection for the point $(0, 0) \in A(0)$ is $a(x) = (0, 0)$ for $x \geq 0$ and $a(x) = (x, 1)$ for $x < 0$, a discontinuous function.

A natural analogue for additivity or superadditivity in the case of maps defined on convex sets is the so-called *midaffinity* or *midconvexity*, respectively, i. e. affinity or convexity with $\theta = \frac{1}{2}$, and, by extension, with any binary fraction $\theta = \frac{k}{2^n}$, $k, n \in \mathbb{N}$, $0 < k < 2^n$. It is easy to obtain an analogue of Proposition 13 in Section 6 for these properties engage the results of Section 13 to obtain conditions for which midaffinity implies affinity.

The examples showing that the Riesz decomposition property is not always necessary for the existence of a linear selection in Theorem 2 presented in Section 2 and that not every linear set-valued map has a linear selection presented in Section 4 are simple but also unsatisfactory because they use the two-point semilinear space \mathcal{S} . It is not clear whether such examples exist for more regular spaces \mathcal{S} and \mathcal{V} , in particular for a cone \mathcal{S} in a linear space and $\mathcal{V} \subset \mathcal{C}(\mathcal{W})$ for a linear space \mathcal{W} .

Theorem 11 and its extensions reveal a fairly simple necessary and sufficient condition to verify whether a superlinear set-valued map has a linear selection with graph passing through a prescribed point in $\mathcal{S} \times \mathcal{W}$, i. e. whether a linear selection on a one-dimensional subspace can be extended to a selection on all \mathcal{S} . Higher order conditions, i. e. when there are several points in $\mathcal{S} \times \mathcal{W}$ to be passed, or a finite-dimensional subspace of \mathcal{S} , are less clear and require further investigation.

Proposition 26 in Section 9 is in many ways unsatisfactory, as it does not tell whether the results about linear selections described in this paper are applicable to sets closed in measure for general Banach ideal lattices and not just those on the measurable space \mathbb{Z} . However, results of Section 12 compensate to a large degree for this deficiency.

Despite Theorem 32, results of Section 10 for semilinear spaces with cone-bases may still be useful. If X is a suitable space of functions on a suitable topological manifold Ω , for example a space of smooth functions $X = C^{(m)}(\mathbb{R}^n)$, $0 \leq m \leq \infty$ or a Lebesgue space $X = L_p(\mathbb{R}^n)$, $0 < p < \infty$, we can construct a cone $K \subset X_+$ that is dense in X_+ and has a cone-basis. This is easily done using the Bernstein polynomials $b_{n,k}(t) = \binom{n}{k} (1-t)^{n-k} t^k$, $t \in [0, 1]$, and a standard partition of unity.

Thus we can construct at no cost a linear selection S for a superlinear map T on K . If continuity of S could be ensured somehow, this would yield a continuous linear selection S of T on all X . Results of Section 11, however, show that this would generally require some additional assumptions imposed on T beyond upper semicontinuity. There is a more direct way to see this subtle point, which is that representation of a function f from $C_{0+}([0, 1])$ (i. e. continuous and bounded away from 0) as a sum $f = \sum_{n,k} \alpha_{n,k} b_{n,k}$, $\alpha_{n,k} > 0$, is *not unique*, as in fact an infinite number of such decompositions are obtained by first choosing $n_1 \in \mathbb{N}$ and a decomposition $f = f_1 + f_2$, $f_1 = \sum_k \alpha_{n_1,k} b_{n_1,k}$, $f_2 \in C_{0+}([0, 1])$ and subsequently decomposing f_2 into a combination of $b_{n,k}$ with $n > n_1$. Therefore arbitrary choice of values of T on $b_{n,k}$ does not guarantee in general that the map S obtained in this manner will be continuous.

Singularities of the functions obtained with the help of Theorem 44 are contained in the set G . One would hypothesize that for any prescribed set $A \subset \mathbb{T}^n$ dense in \mathbb{T}^n a suitable measure supported in A can be constructed. Results similar to Theorem 44 are likely possible for the unit ball of \mathbb{C}^n and general strictly pseudoconvex domains; see, e. g., [34]. This would also make it possible to nearly effortlessly obtain extensions of Theorem 46 for such domains, as the corresponding analogues of Theorem 45 have been available for some time; see, e. g., [2], [3].

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